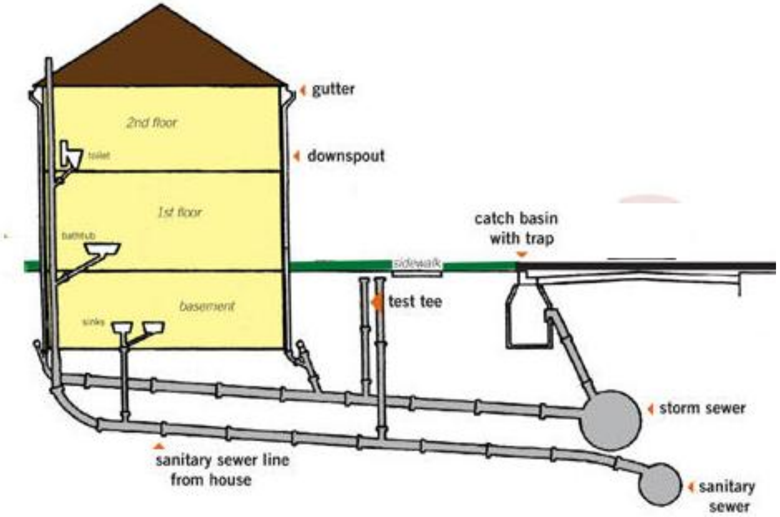
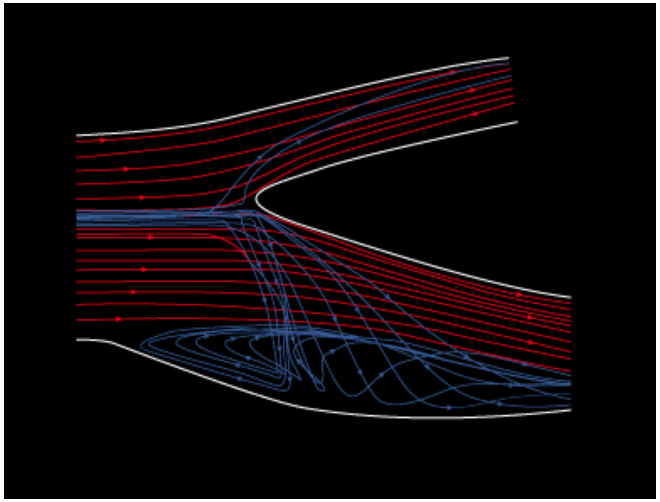
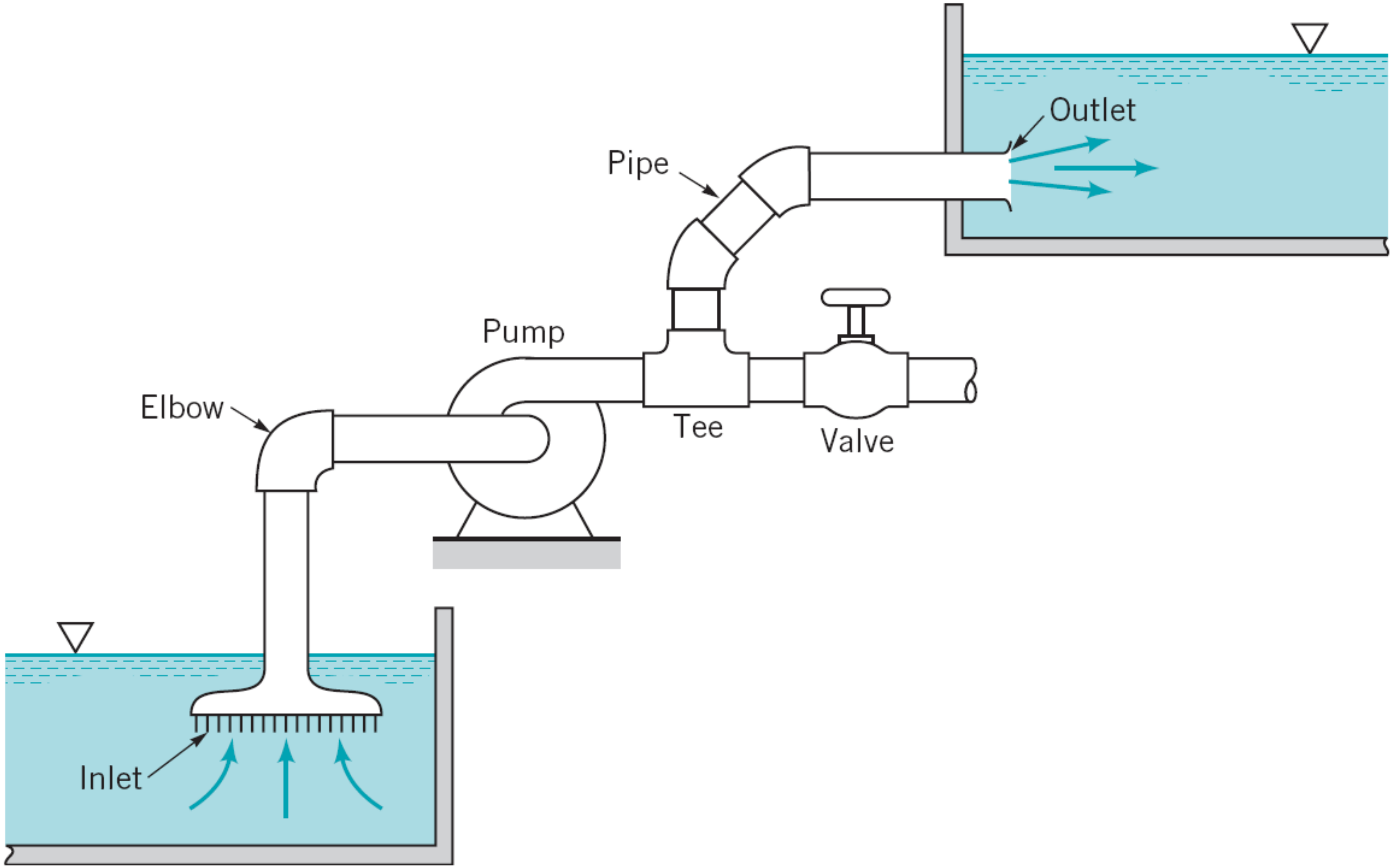
Viscous flow in pipes



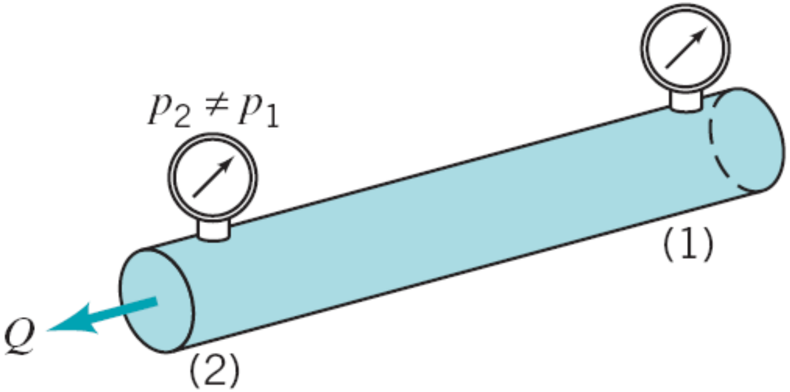


Pipe flow is a crucial aspect of fluid mechanics

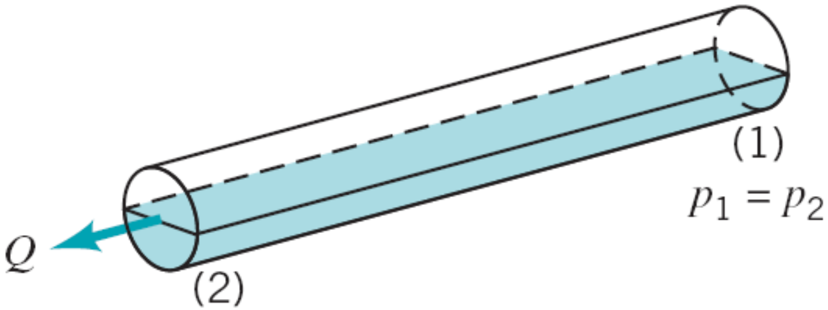
Typical pipe system components.



Pipe flow (pressure-driven)

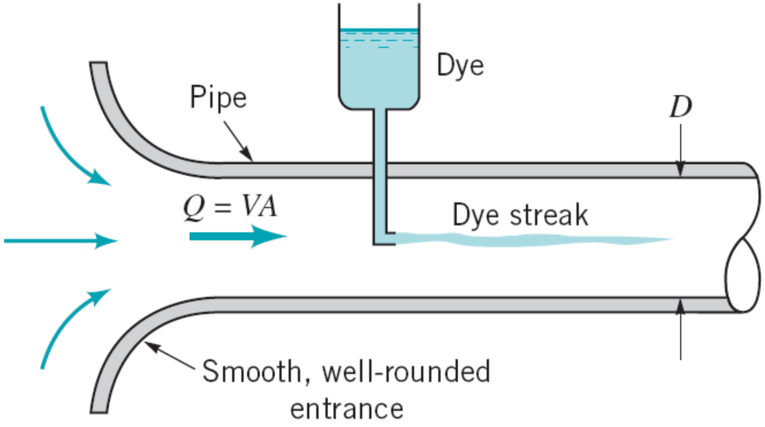


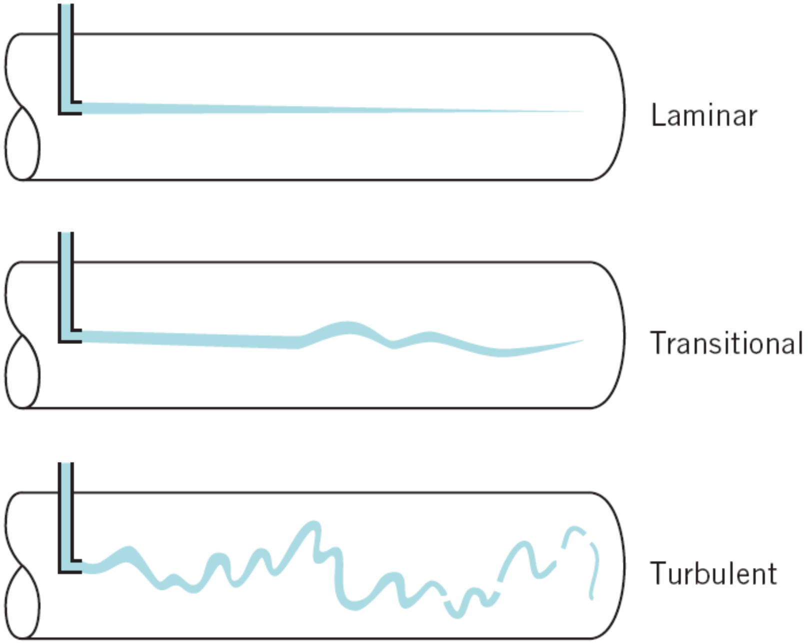
Channel flow (gravity-driven)



Laminar, transitional and turbulent flow

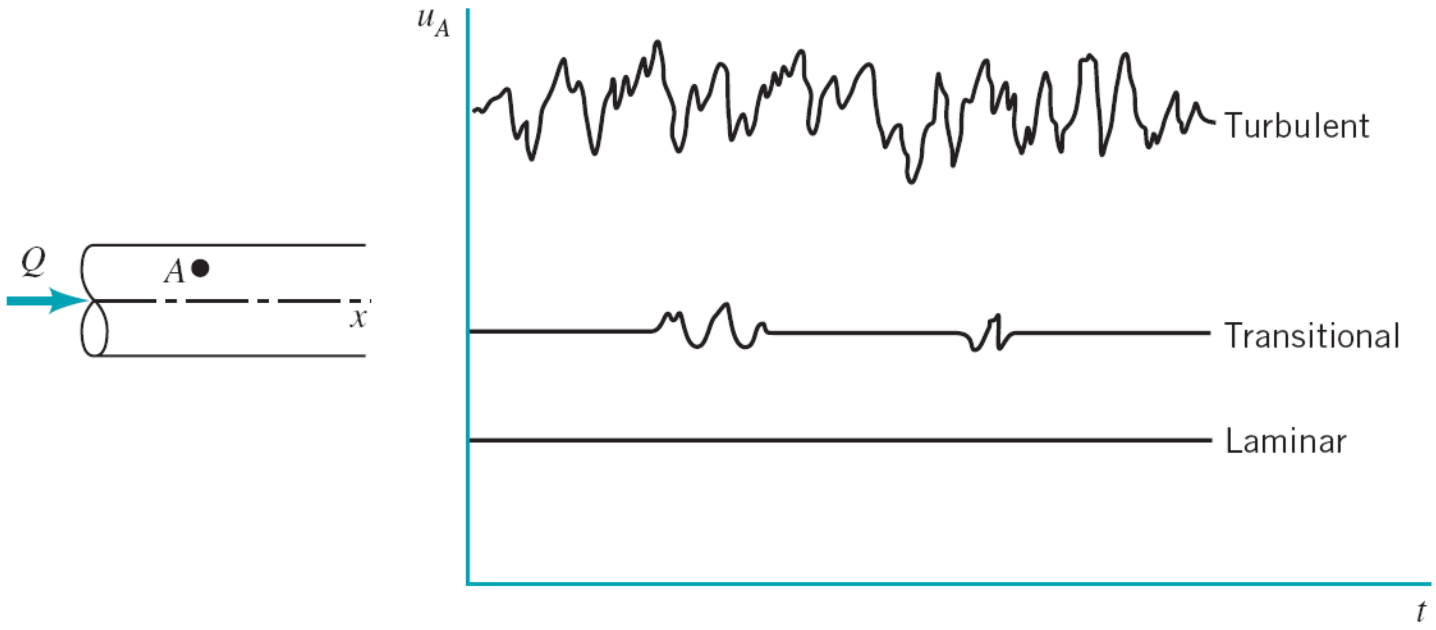
Reynolds experiment:





Laminar, transitional and turbulent flow (video)

Limits of laminar, transitional and turbulent flow



Re > 4000

Re < 2100

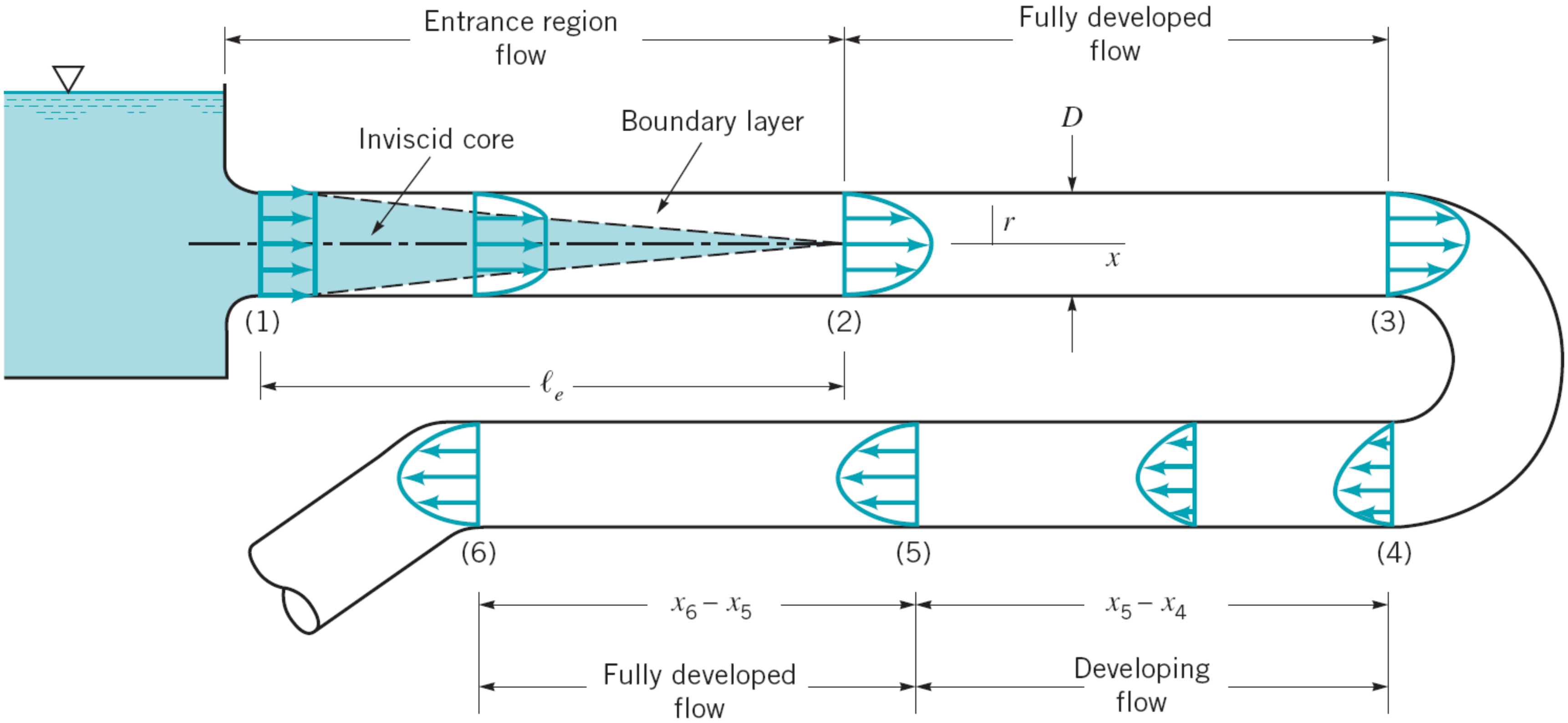
Remarks:

a) Analytical solutions only for laminar flow

b) Only few engineering applications involve laminar flow. Most involve turbulent flow. For turbulent flow we use experimental findings and empirical relations

c) Before using Poiseuille's law (valid only for laminar flow), check the Reynolds number to verify that the flow is indeed laminar (Re < 2100)

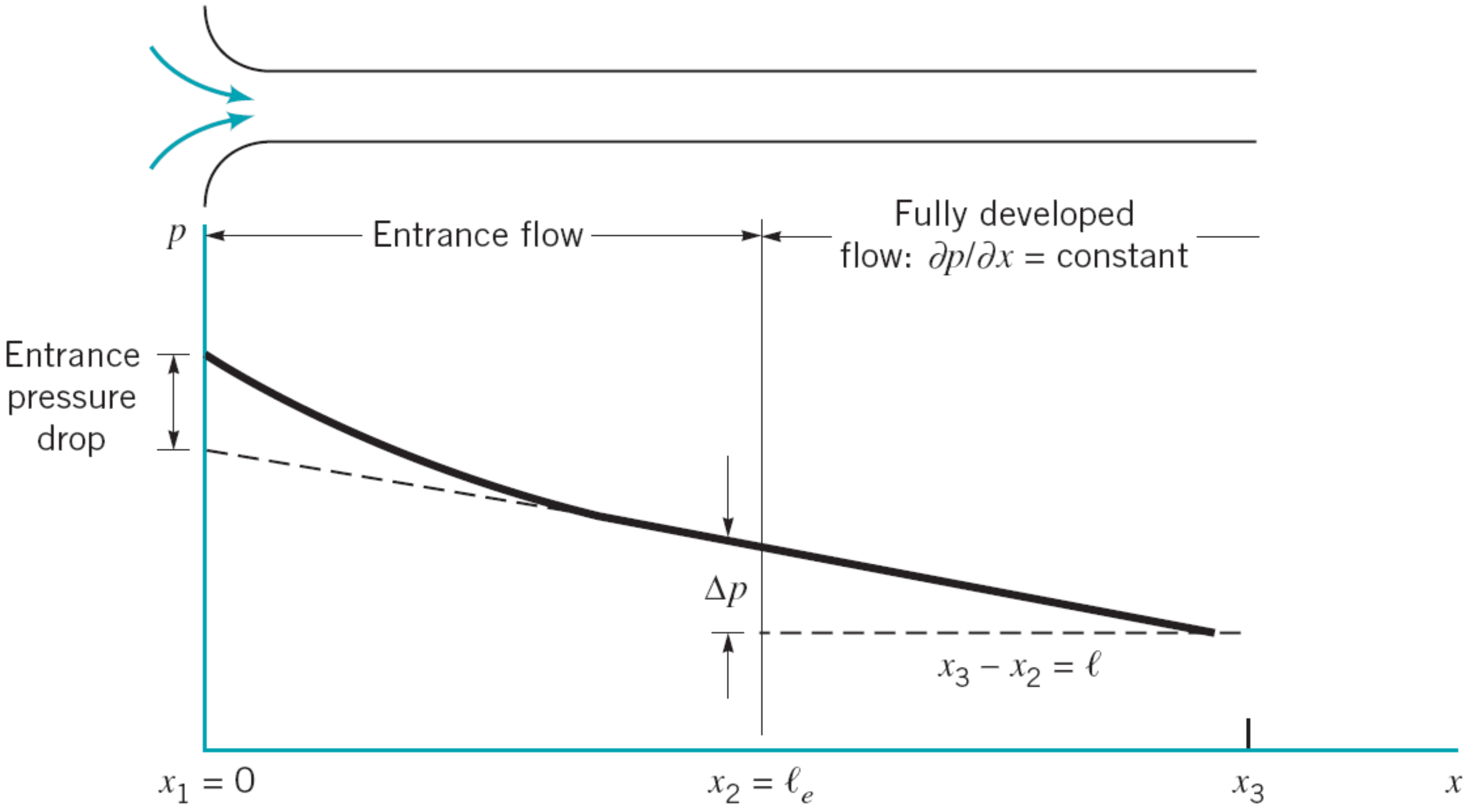
Entrance region, developing flow, and fully developed flow



$$\frac{l_e}{D} = 0.06\text{Re}$$ For laminar flow

$$\frac{l_e}{D} = 4.4\text{Re}^{1/6}$$ For turbulent flow

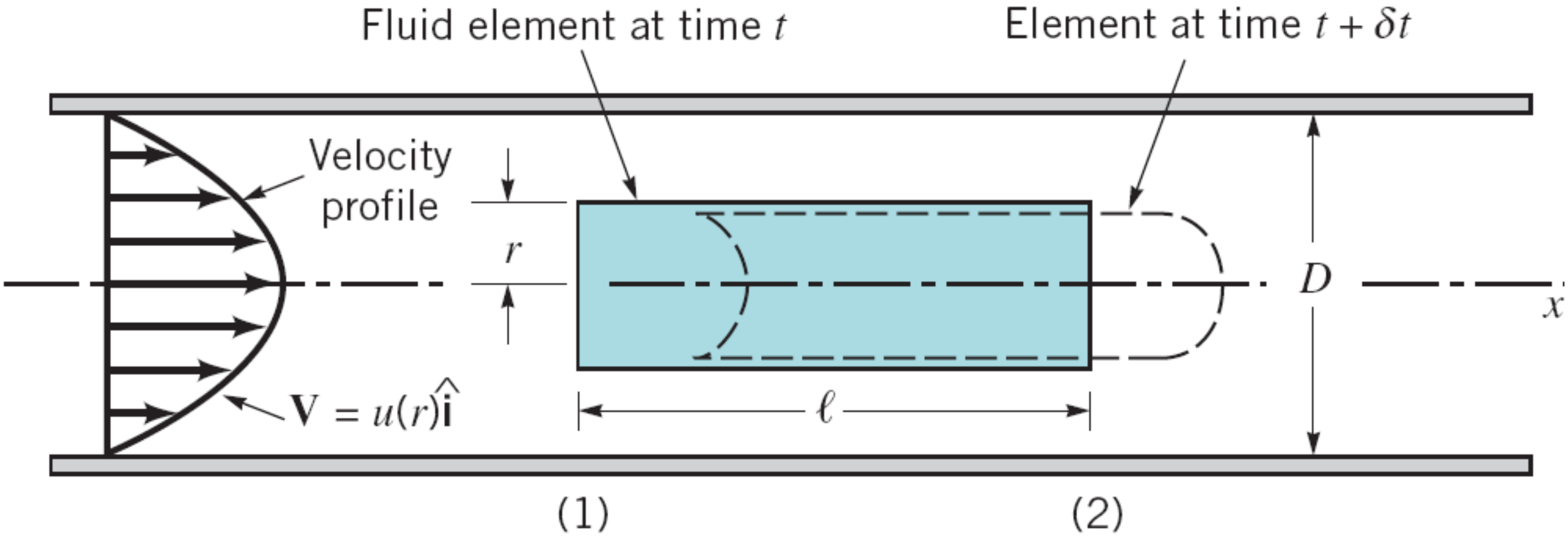
Pressure distribution along a horizontal pipe.

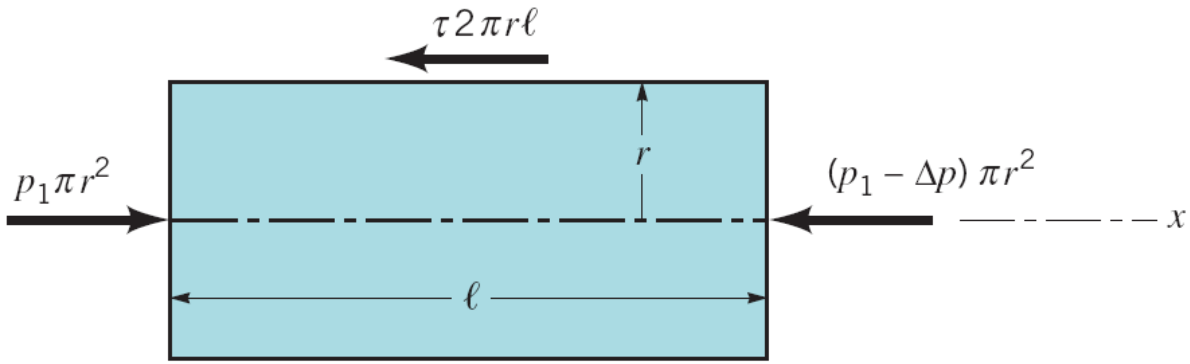


In fully developed region: $\frac{\Delta p}{l} = -\frac{\partial p}{\partial x} = \text{constant}$

no accelerations: $a_x = 0$

Fully developed laminar flow





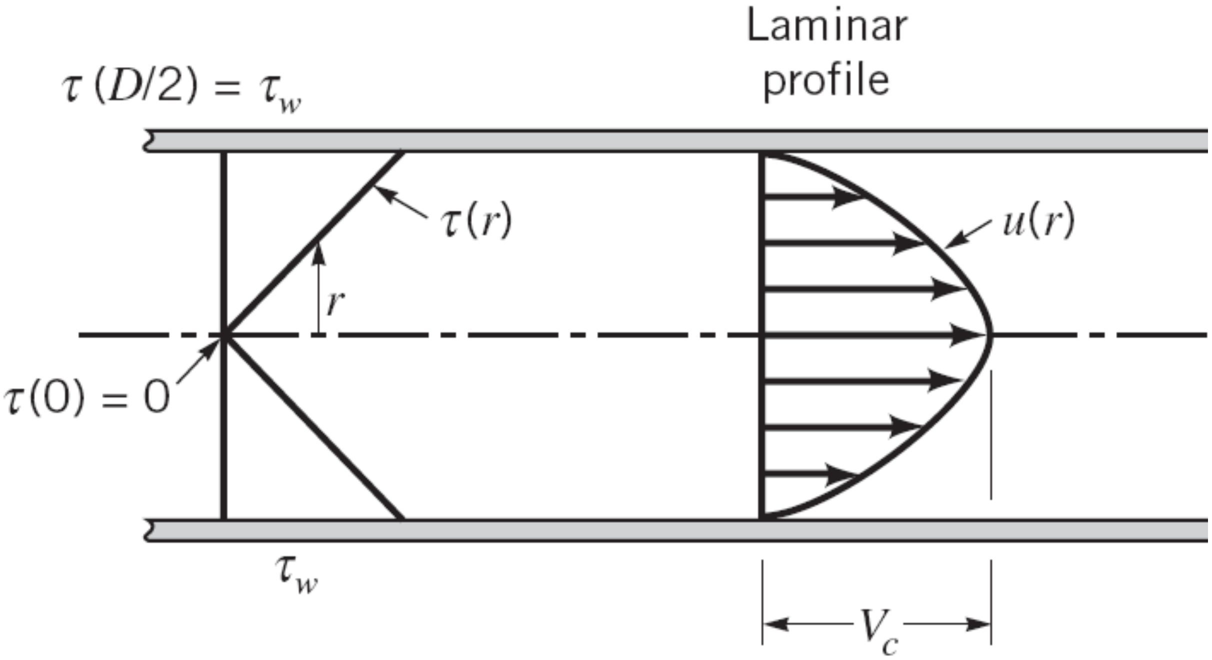
Free-body diagram of a cylindrical fluid element

$$\sum F_x = 0 \Rightarrow p \cdot \pi r^2 - (p - \Delta p)\pi r^2 - \tau\, 2\pi r \ell = 0$$

$$\Rightarrow \quad \Delta p = \frac{2\tau\ell}{r}$$

$$\Rightarrow \quad \frac{\Delta p}{\ell} = \frac{2\tau}{r} \qquad \text{(independent of } r\text{)}$$

Shear stress distribution within the fluid in a pipe



Note: a) @ $r=0$ $\tau=0$ no shear @ centerline

b) @ $r=R$ $\tau=\tau_w = \frac{\Delta p}{\ell}\frac{R}{2}$ maximum shear

Relation of pressure drop to velocity and flow
Hagen-Poiseuille law

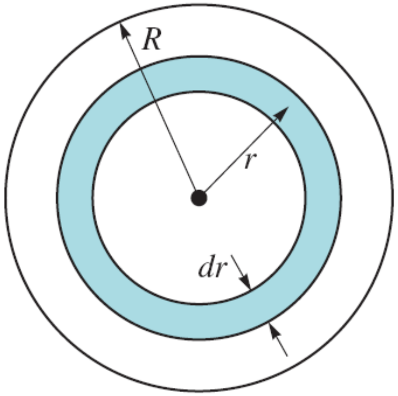
$$\left.\begin{aligned} \tau &= -\mu \frac{du}{dr} \\ \tau &= \frac{\Delta p}{\ell}\frac{r}{2} \end{aligned}\right\} \Rightarrow \frac{du}{dr} = -\frac{\Delta p}{2\mu\ell} r$$

$$\Rightarrow \int du = -\frac{\Delta p}{2\mu\ell}\int r\,dr \quad \Rightarrow \quad u = -\frac{\Delta p}{4\mu\ell} r^2 + C$$

B.C.: At $r = R$ $\quad u = 0 \Rightarrow C = \frac{\Delta p}{4\mu\ell} R^2$

$$\therefore \quad \underline{\underline{u = \frac{\Delta p}{4\mu\ell}\left[R^2 - r^2\right]}} \quad \text{or} \quad \underline{\underline{u = \frac{\Delta p R^2}{4\mu\ell}\left[1 - \frac{r^2}{R^2}\right]}}$$

At $r = 0$ $\quad u = V_c = \frac{\Delta p R^2}{4\mu\ell}$ $\quad$ Hence $\quad u(r) = V_c \cdot \left[1 - \frac{r^2}{R^2}\right]$



$dA = 2\pi r\,dr$

To get flow, Q, integrate over the cross section

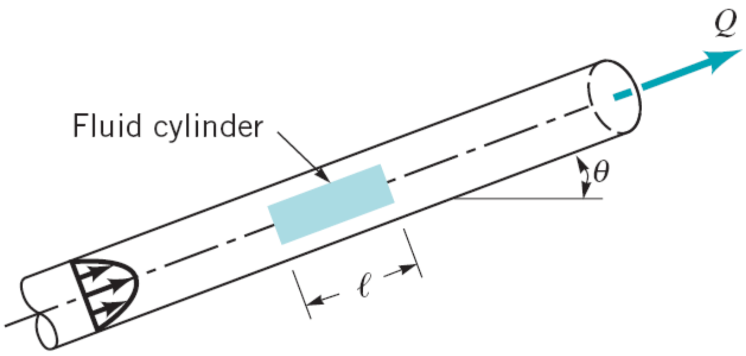
$$Q = \int_0^R u(r)\, 2\pi r\, dr = 2\pi V_c \int_0^R r\left[1 - \frac{r^2}{R^2}\right] dr$$

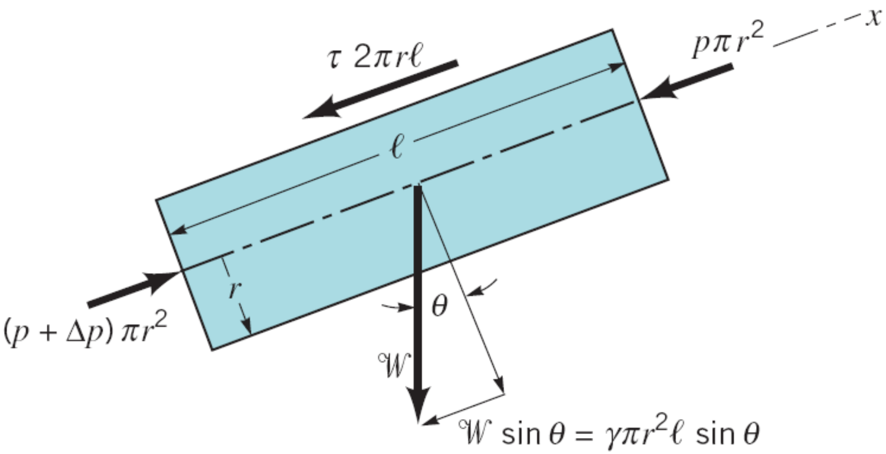
$$\Rightarrow \quad Q = 2\pi V_c \int_0^R \left[r - \frac{r^3}{R^2}\right] dr = 2\pi V_c \left[\frac{r^2}{2} - \frac{r^4}{4R^2}\right]_0^R$$

$$\Rightarrow \quad \underline{\underline{Q = \frac{\pi R^2 V_c}{2}}} \qquad \text{or} \qquad \underline{\underline{Q = \frac{\pi R^4}{8\mu l}\Delta p}}$$

Average velocity: $\overline{V} = \dfrac{Q}{\pi R^2} = \dfrac{V_c}{2}$

Laminar flow in non-horizontal pipes





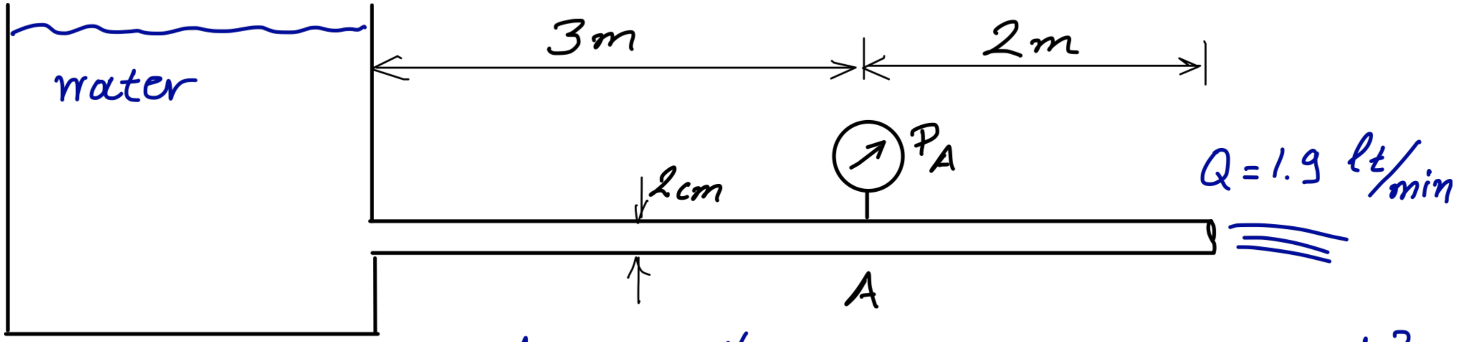
$$\Sigma F_x = 0 \Rightarrow \Delta p \cdot \pi r^2 - \gamma \pi r^2 \ell \sin\theta - 2\pi r \ell \cdot \tau = 0$$

$$\Rightarrow (\Delta p - \gamma \ell \sin\theta)\pi r^2 - \tau\, 2\pi r \ell = 0$$

∴ In Poiseuille's law, replace Δp with $\Delta p - \gamma \ell \sin\theta$

i.e.,
$$Q = \frac{\pi R^4 (\Delta p - \gamma \ell \sin\theta)}{8 \mu \ell}$$

Example



What is the pressure at point A?

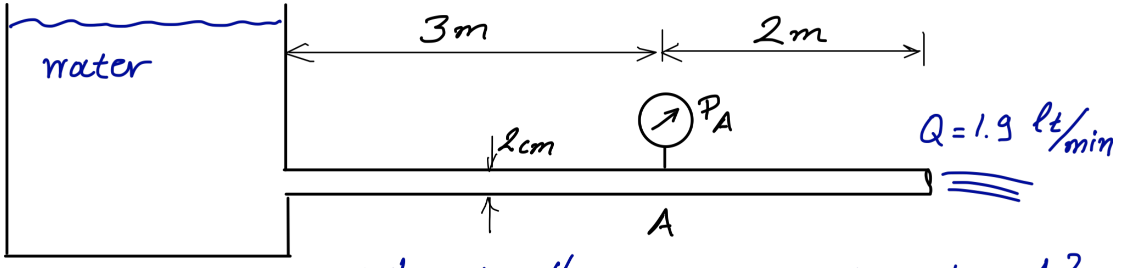
$$Q = 1.9\ \text{lt}/\text{min} = \frac{1.9 \times 10^{-3}\ m^3}{60\ s} = 3.17 \times 10^{-5}\ m^3/s$$

$$V = \frac{Q}{A} = \frac{3.17 \times 10^{-5}\ m^3/s}{\pi\ 0.01^2\ m^2} = 0.101\ m/s$$

$$Re = \frac{\rho V D}{\mu} = \frac{999 \times 0.101 \times 0.02}{1.12 \times 10^{-3}} = 1800$$

$Re = 1'800 < 2'100$ flow is laminar

Example



What is the pressure at point A?

$$Q = \frac{\pi R^4}{8\mu l}\Delta p \Rightarrow \Delta p = \frac{8\mu l}{\pi R^4} Q \qquad \left(\Delta p = P_A - \cancel{P_{exit}} \rightarrow 0\right)$$

$$\Rightarrow \Delta p = \frac{8 \times 1.12\times10^{-3} \times 2}{\pi \cdot 0.01^4} \times 3.17 \times 10^{-5}\ N/m^2 = \underline{\underline{18.1\ Pa}}$$

$$\Rightarrow \underline{\underline{P_A = 18.1\ Pa}}$$

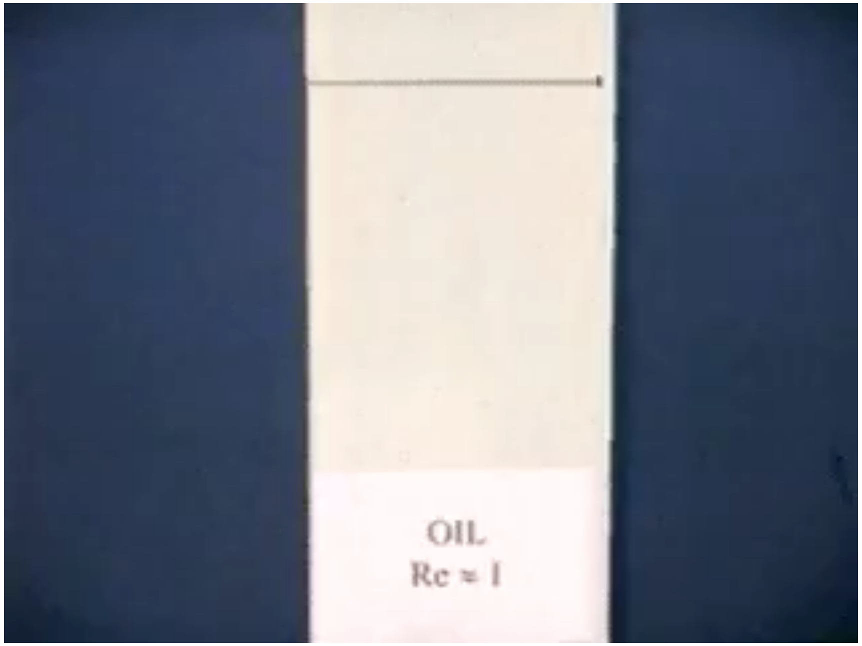
Need to check if flow is fully developed at A:

$$l_e = 0.06 \times Re \times D = 0.06 \times 1800 \times 0.02\ m = 2.16\ m$$

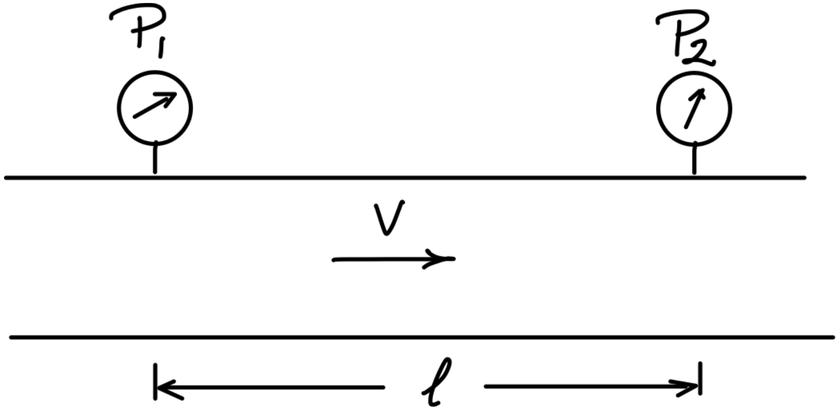
$l_e = 2.16\ m < 3\ m$ so flow is fully developed at A.

OK to use Poiseuille.

Laminar & turbulent flow



Energy equation applied to pipe flow

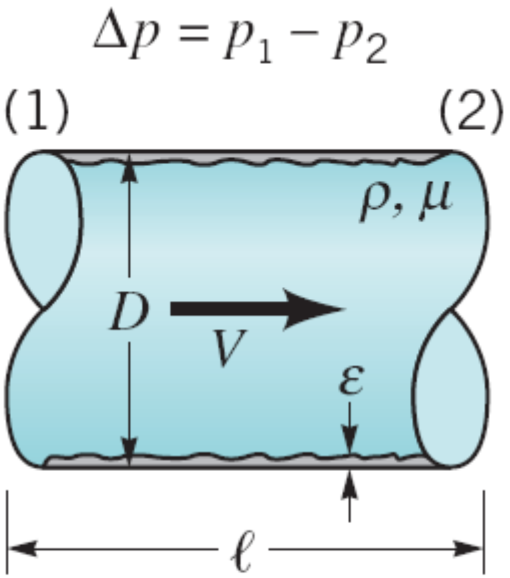


Energy: $$\frac{P_1}{\gamma} + \alpha_1 \frac{V_1^2}{2g} + z_1 = \frac{P_2}{\gamma} + \alpha_2 \frac{V_2^2}{2g} + z_2 + h_L$$

Velocity profile is the same: $$\alpha_1 \frac{V_1^2}{2g} = \alpha_2 \frac{V_2^2}{2g}$$

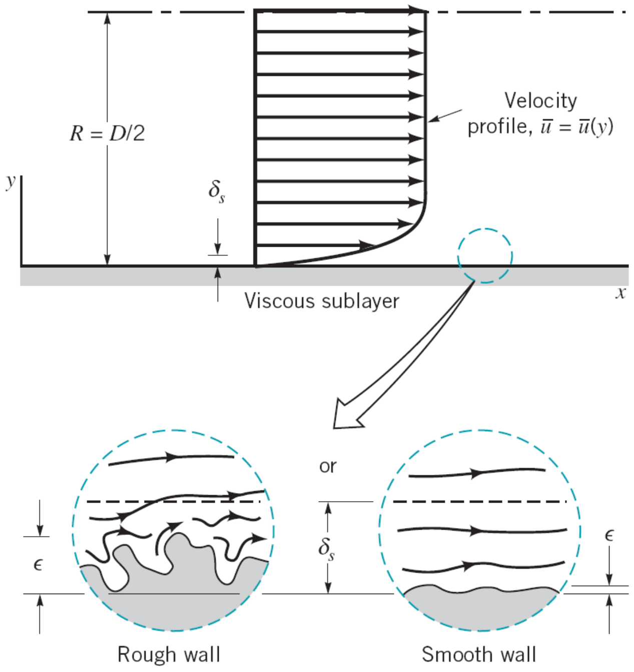
Hence, $$h_L = \frac{P_1 - P_2}{\gamma} \qquad (1)$$

Dimensional analysis



ε = pipe wall roughness

Typically, 0 ≤ ε ≤ 0.05



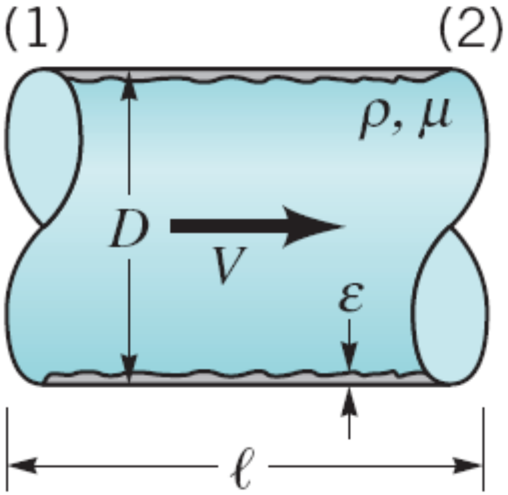
Flow in the viscous sublayer near rough and smooth walls.

$$\Delta p = f(D, \ell, \varepsilon, \mu, \rho, V)$$

Π- theorm: $$\frac{\Delta p}{\frac{1}{2}\rho V^2} = \phi\left(\frac{\ell}{D}, \frac{\varepsilon}{D}, \frac{\rho V D}{\mu}\right)$$

Dimensional analysis

$\Delta p = p_1 - p_2$



$$\frac{\Delta p}{\frac{1}{2}\rho V^2} = \phi\left(\frac{\ell}{D}, \frac{\varepsilon}{D}, \frac{\rho V D}{\mu}\right)$$

Assume $\Delta p \sim \ell$

$$\frac{\Delta p}{\frac{1}{2}\rho V^2} = \frac{\ell}{D} \cdot \underbrace{\phi\left(\frac{\varepsilon}{D}, Re\right)}_{\text{friction factor, } f}$$

$$h_L = \frac{P_1 - P_2}{\gamma} \quad (1) \qquad \therefore \quad \Delta p = f \cdot \frac{\ell}{D} \cdot \frac{\rho V^2}{2} \quad (2)$$

$$(1) \wedge (2) \Longrightarrow \quad h_L = f \cdot \frac{\ell}{D} \cdot \frac{V^2}{2g}$$

Friction factor for laminar flow

$$\Delta P = \frac{8\mu l}{\pi R^4} Q = \frac{8\mu l}{R^2}\frac{Q}{\pi R^2} = \frac{8\mu l}{R^2} V = \frac{32\mu l}{D^2} V$$

$$\Rightarrow \quad \Delta p = 32 \frac{l}{D} \cdot \frac{\mu}{\rho V D} \cdot \rho V^2$$

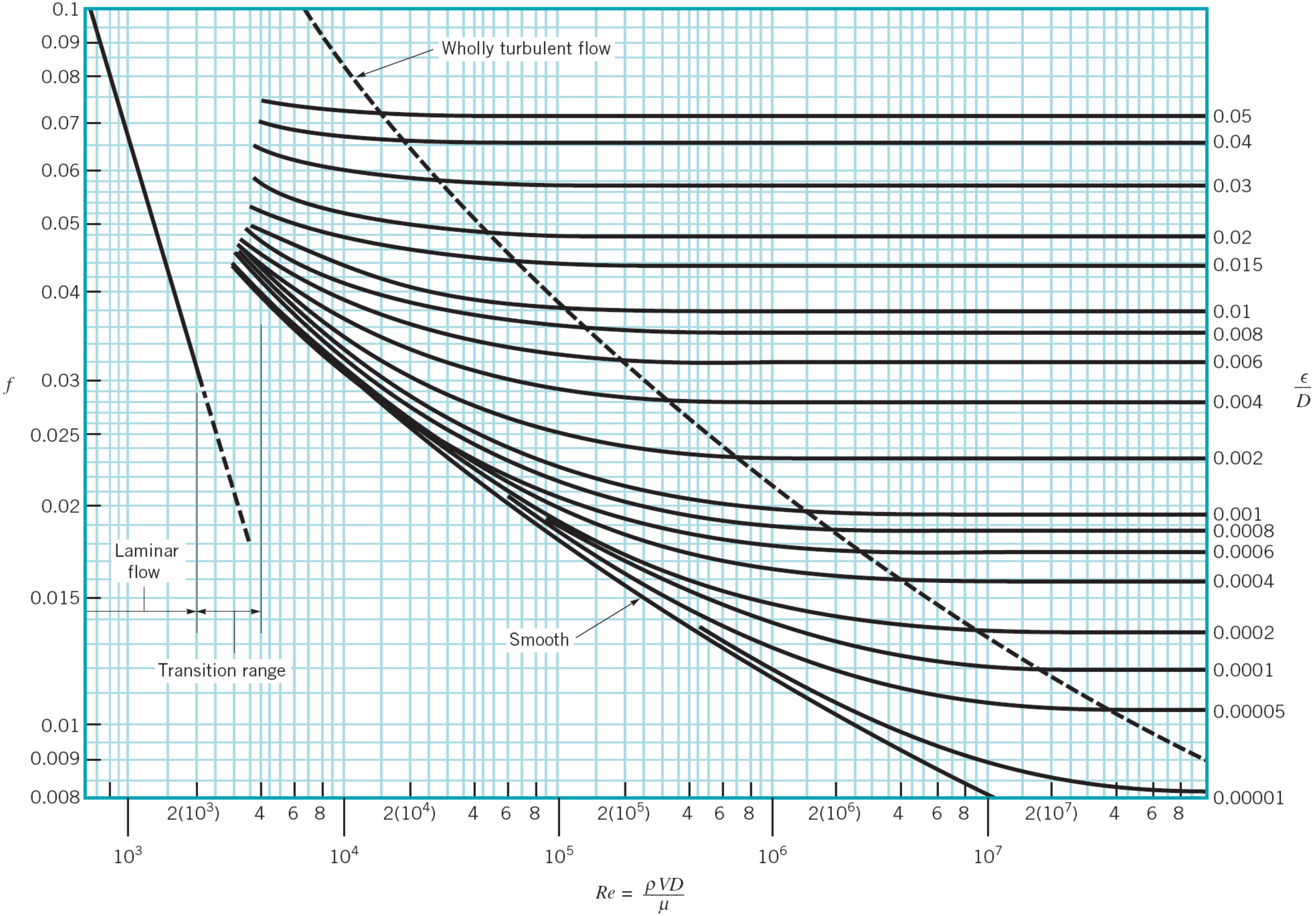
$$\Rightarrow \quad \Delta p = \underbrace{\frac{64}{Re}}_{f} \frac{l}{D} \frac{\rho V^2}{2}$$

$$\Rightarrow \quad \underline{\underline{f = \frac{64}{Re}}}$$

(Independent of wall roughness)

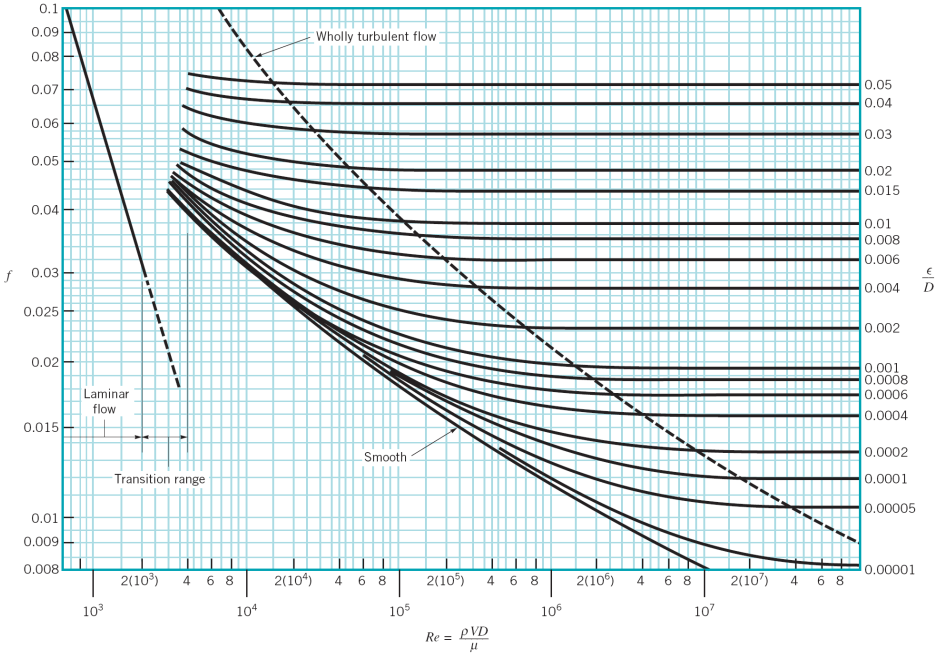
Friction factor as a function of Reynolds number and relative roughness

Moody chart



Colebrook’s formula

Moody’s chart



Colebrook’s formula

Empirical formula, equivalent to Moody’s chart:

$$\frac{1}{\sqrt{f}} = -2.0\log\left(\frac{\varepsilon/D}{3.7} + \frac{2.51}{\mathrm{Re}\sqrt{f}}\right)$$

To solve, use iterative method until convergence. For example, if Re and f are known, assume a friction factor value, plug it on RHS of the equation to obtain a new f value on the LHS. Repeat the procedure with the new f value until convergence.

Equivalent Roughness for New Pipes

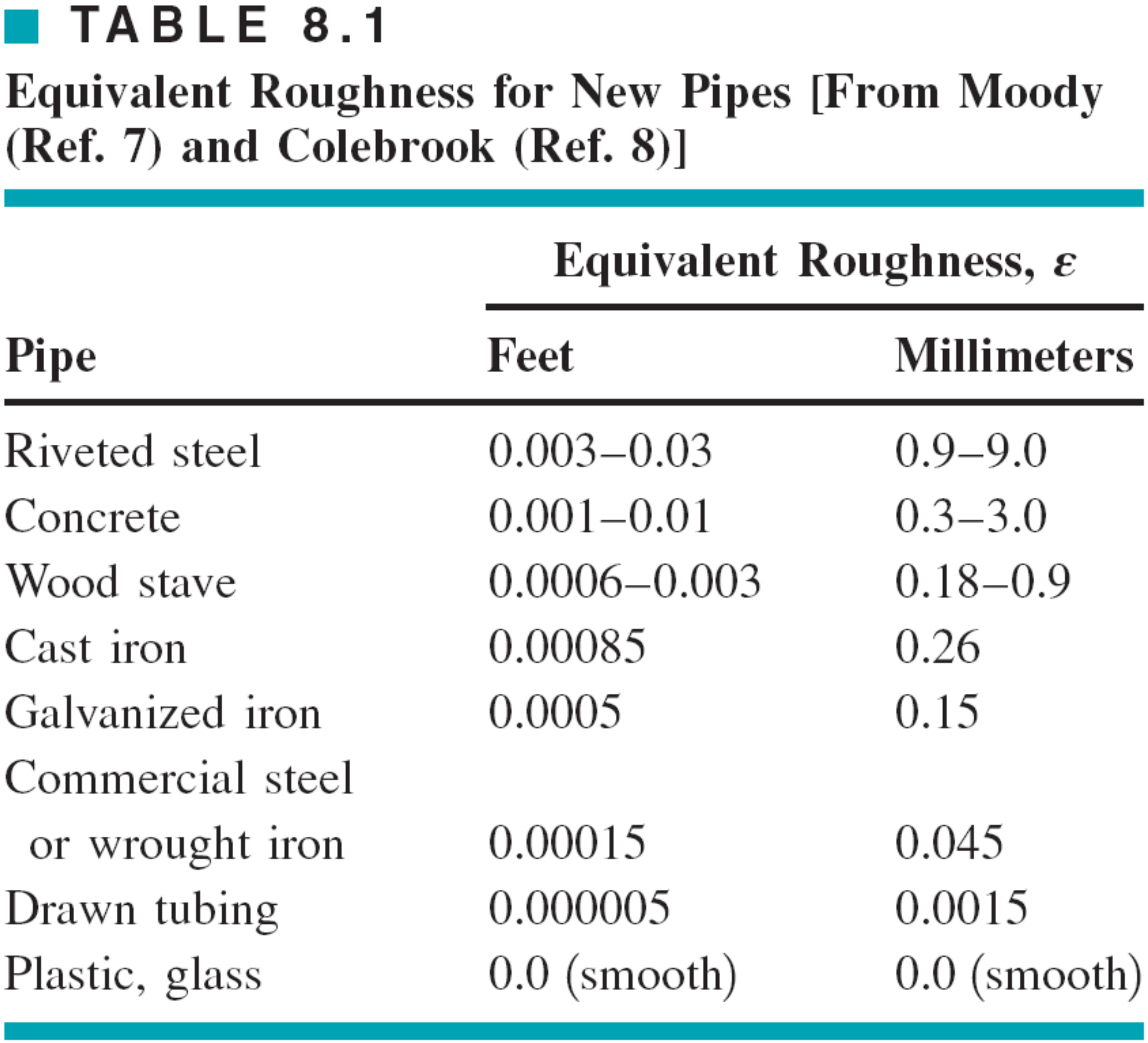
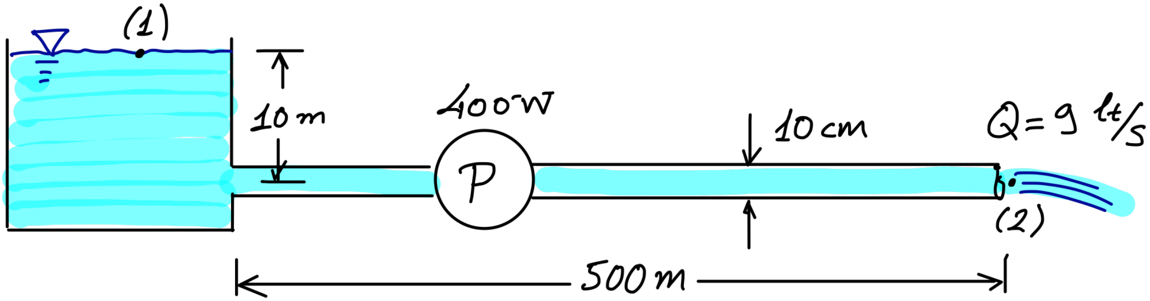


TABLE 8.1

Equivalent Roughness for New Pipes [From Moody (Ref. 7) and Colebrook (Ref. 8)]

	Equivalent Roughness, ε	
Pipe	**Feet**	**Millimeters**
Riveted steel	0.003–0.03	0.9–9.0
Concrete	0.001–0.01	0.3–3.0
Wood stave	0.0006–0.003	0.18–0.9
Cast iron	0.00085	0.26
Galvanized iron	0.0005	0.15
Commercial steel or wrought iron	0.00015	0.045
Drawn tubing	0.000005	0.0015
Plastic, glass	0.0 (smooth)	0.0 (smooth)

Example



What is the pipe roughness?

Energy equation between (1) and (2):

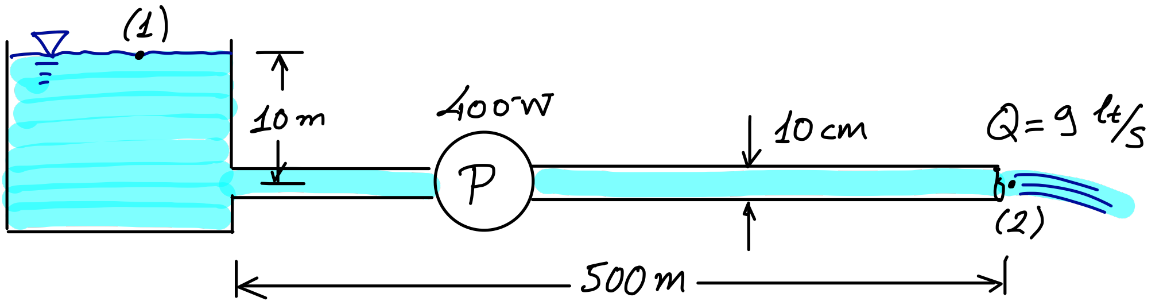
$$\cancel{\frac{P_1}{\gamma}}^{\to 0} + \cancel{\frac{V_1^2}{2g}}^{\to 0} + z_1 + h_p = \cancel{\frac{P_2}{\gamma}}^{\to 0} + \frac{V_2^2}{2g} + z_2 + h_L$$

$$\Rightarrow \quad h_L = h_p + (z_1 - z_2) - \frac{V_2^2}{2g}$$

$$h_p = \frac{\dot{W}}{\gamma Q} = \frac{400 \ N.m/s}{9800 \ N/m^3 \cdot 0.009 \ m^3/s} = 4.54 \ m$$

$$V_2 = \frac{Q}{A} = \frac{0.009 \ m^3/s}{\pi \cdot 0.05^2 m^2} = 1.15 \ m/s$$

Example

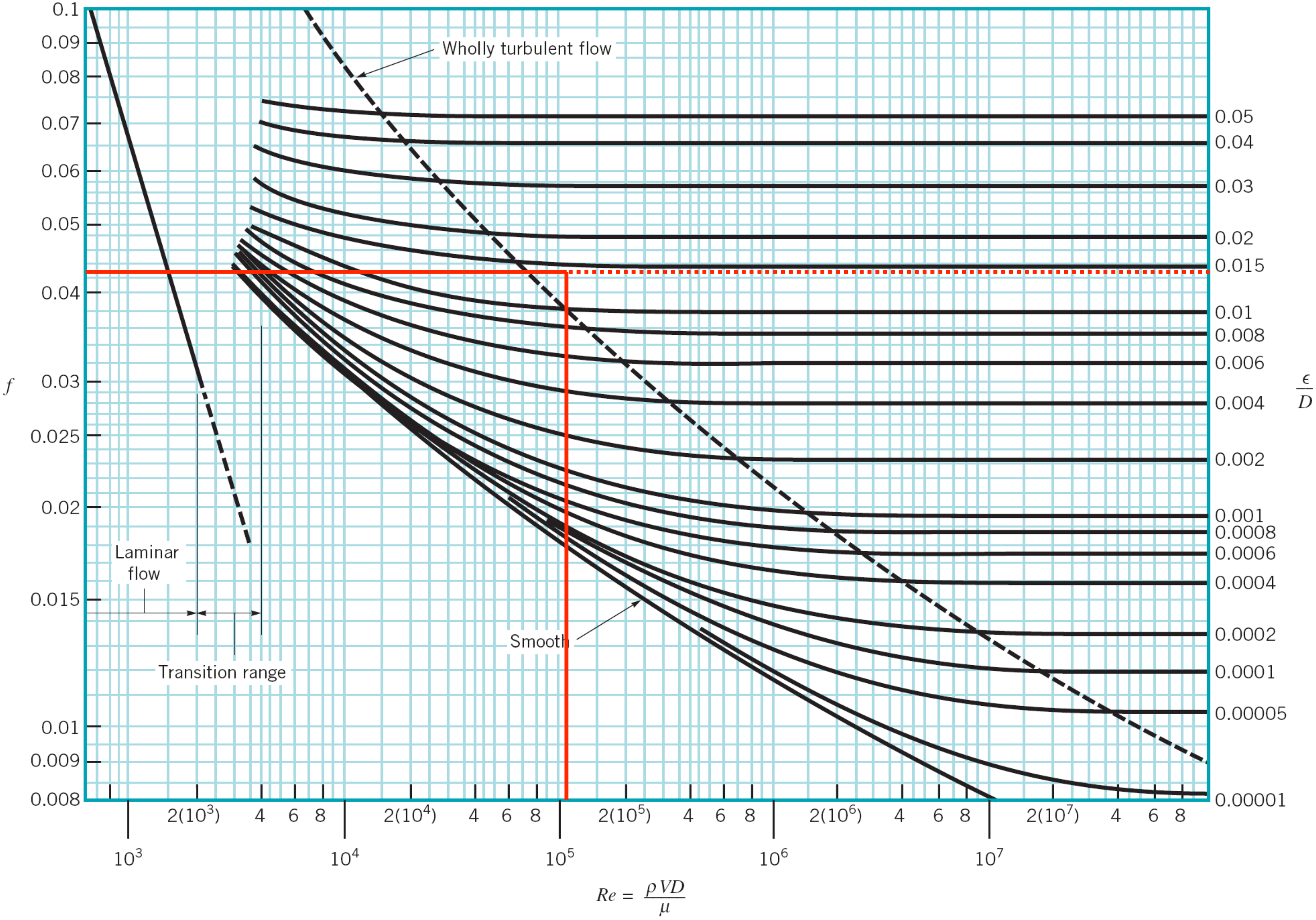


$$\therefore\ h_L = 4.54\ m + 10\ m - \frac{1.15^2\ m^2/s^2}{2\times 9.81\ m/s^2} = 14.5\,m$$

$$h_L = f\cdot\frac{\ell}{D}\cdot\frac{V^2}{2g} \Rightarrow f = \frac{h_L}{\frac{\ell}{D}\cdot\frac{V^2}{2g}} = \frac{14.5}{\frac{500}{0.1}\cdot\frac{1.15^2}{2\times 9.81}}$$

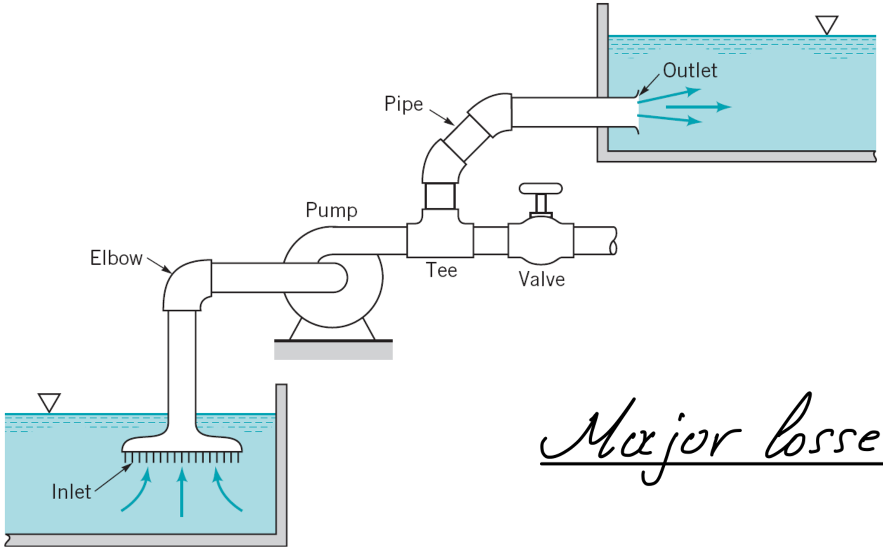
$$\Rightarrow\ \underline{\underline{f = 0.043}}$$

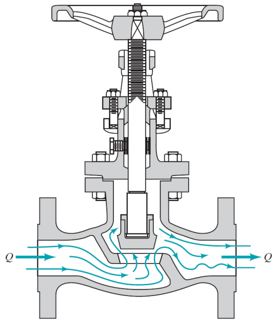
$$Re = \frac{\rho VD}{\mu} = \frac{999\cdot 1.15\cdot 0.1}{1.12\times 10^{-3}} = 1.03\times 10^{-5}$$



$$\frac{\varepsilon}{D} = 0.0145 \Rightarrow \varepsilon = 0.0145 \times 10 \text{ cm} = 0.145 \text{ cm} = 1.45 \text{ mm}$$

Minor losses





Major losses friction losses in straight portions

$$h_L = f \frac{l}{D} \frac{V^2}{2g}$$

Minor losses losses in valves, elbows, tees, etc.
The head loss associated with minor losses:

$$h_L = K_L \frac{V^2}{2g}$$

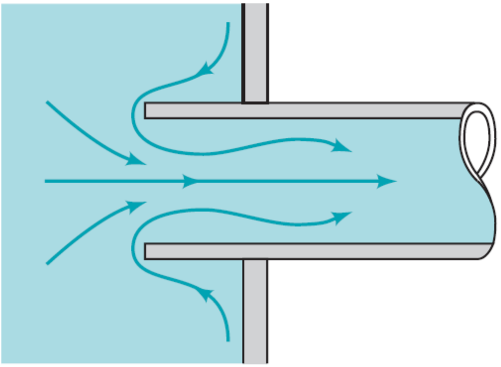
where K_L is function of geometry

$$\Delta p = \gamma \cdot h_L = K_L \frac{\rho V^2}{2}$$

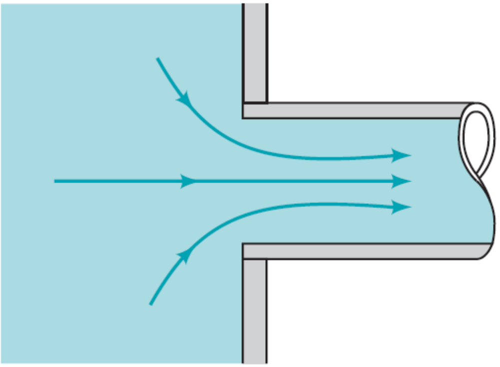
Note:
Minor losses can be more important than major losses.

Entrance flow conditions and loss coefficient

Reentrant
$K_L = 0.8$



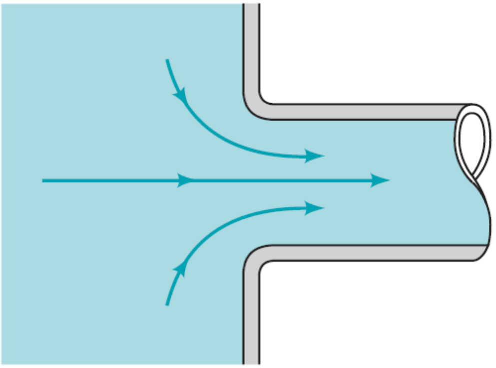
(a)



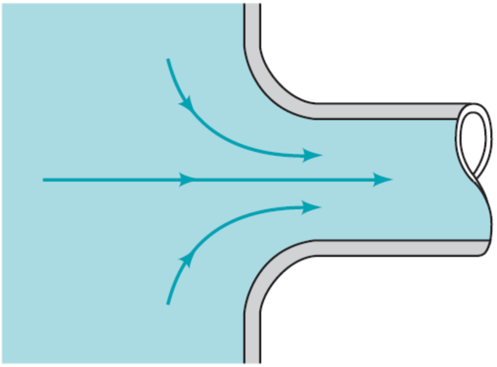
(b)

sharp-edged
$K_L = 0.5$

slightly rounded
$K_L = 0.2$



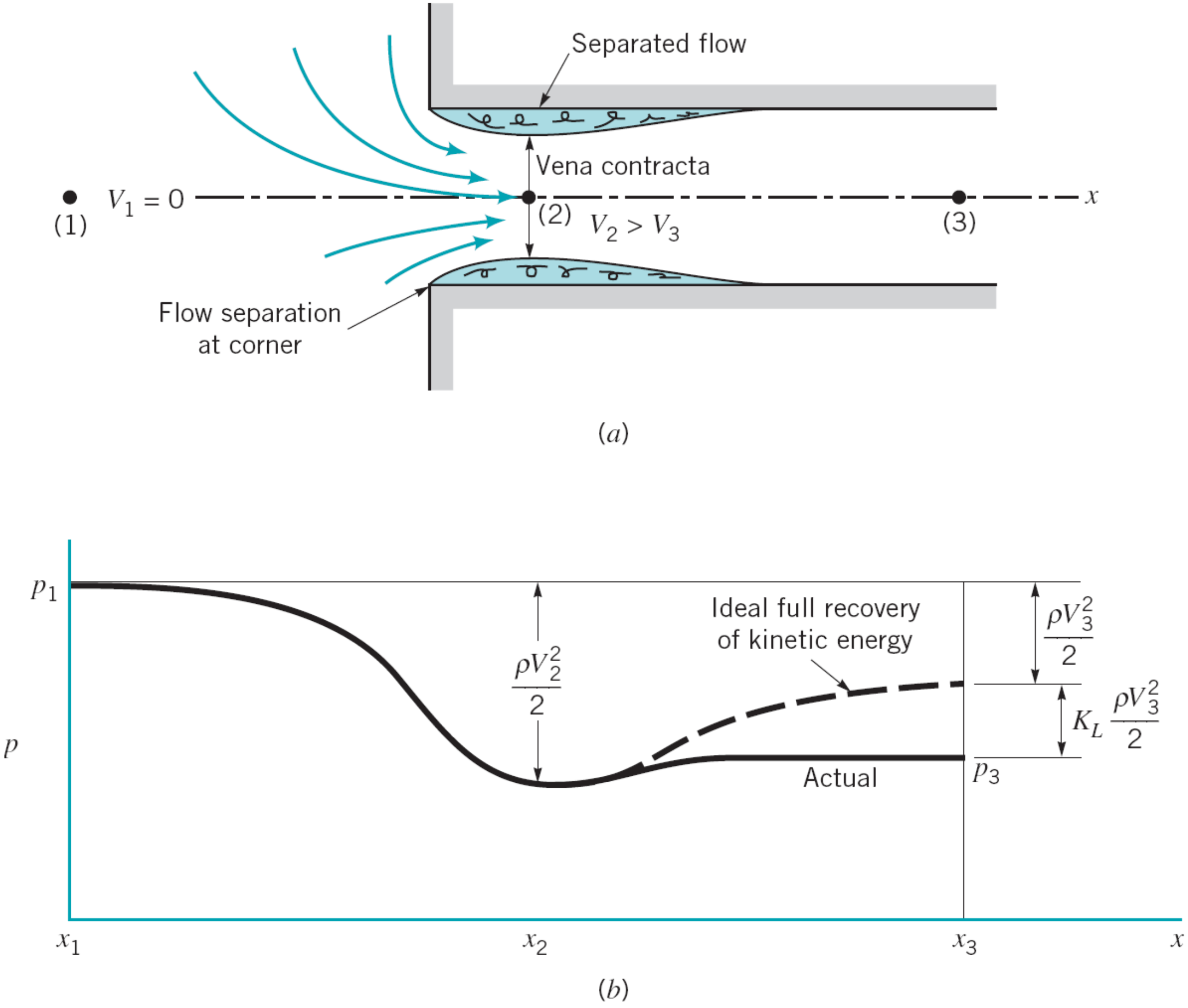
(c)



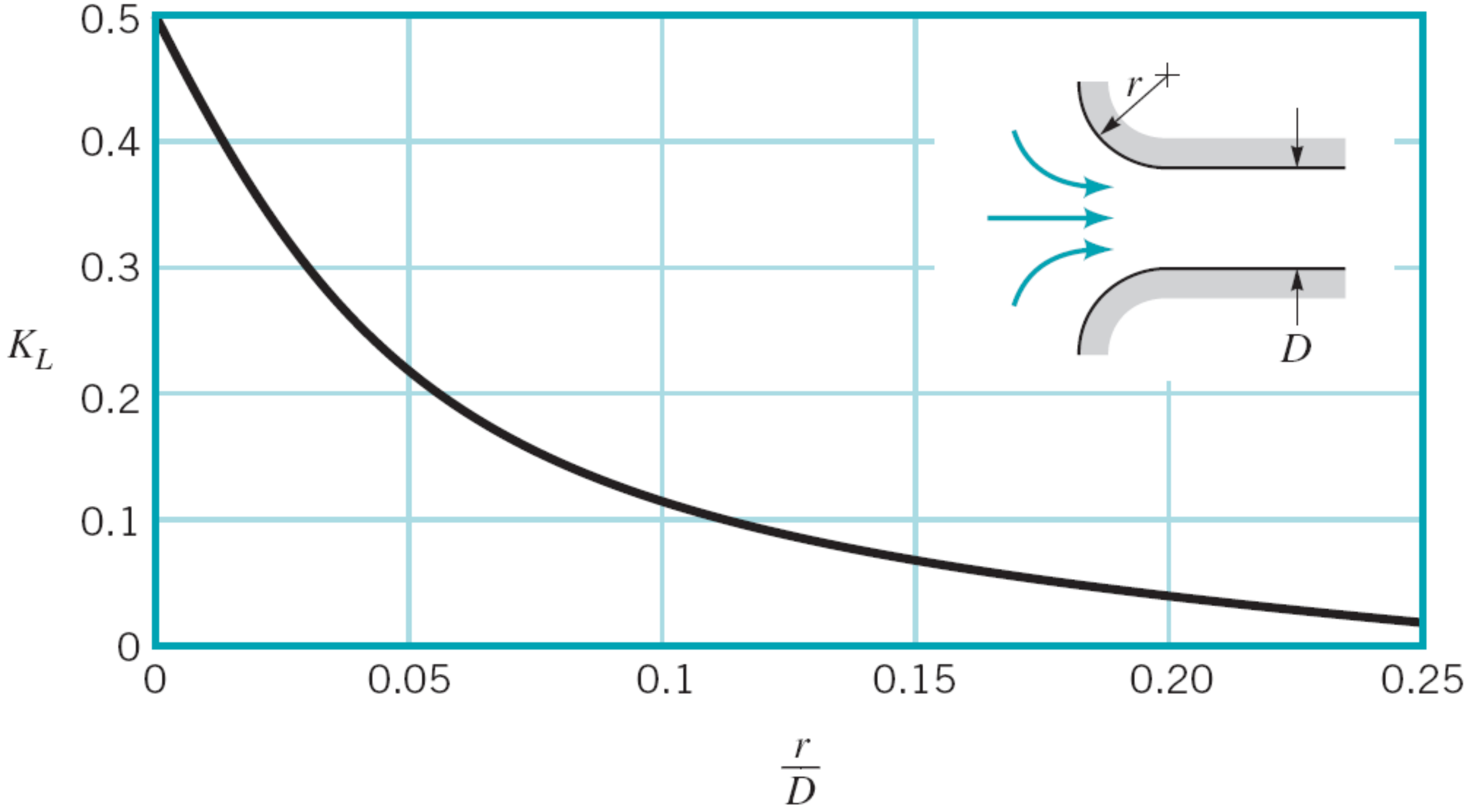
(d)

well-rounded
$K_L = 0.04$

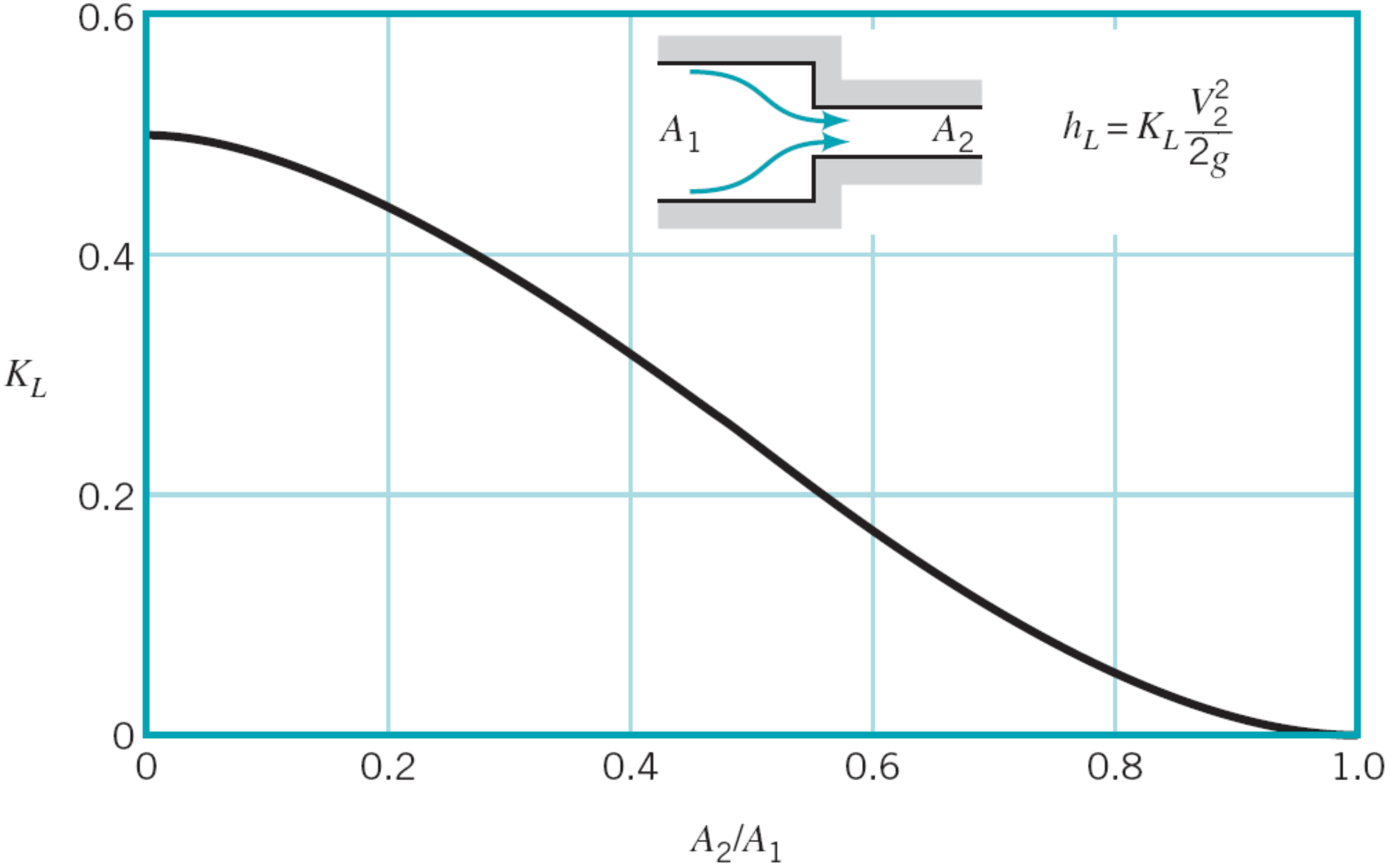
Flow pattern and pressure distribution for a sharp-edged entrance.



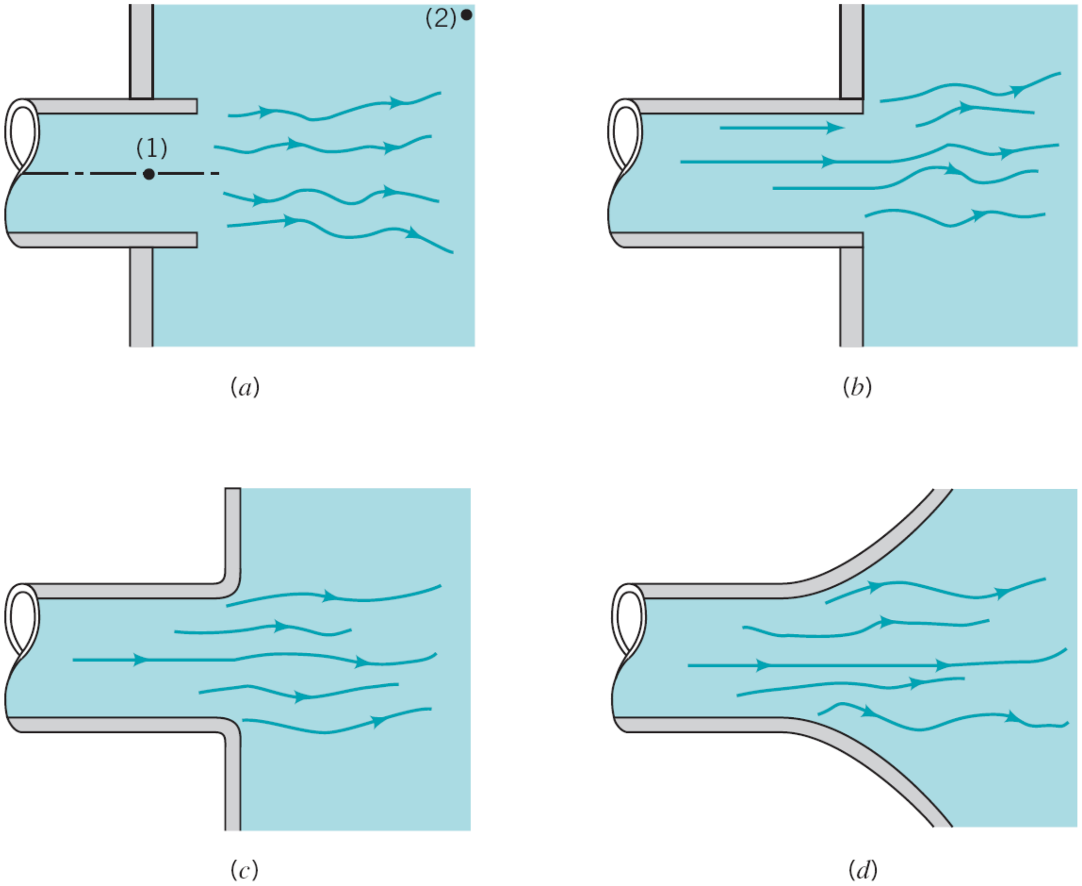
Entrance loss coefficient as a function of rounding of the inlet edge



Loss coefficient for a sudden contraction



Exit flow conditions and loss coefficient



Reentrant
K_L = 1.0

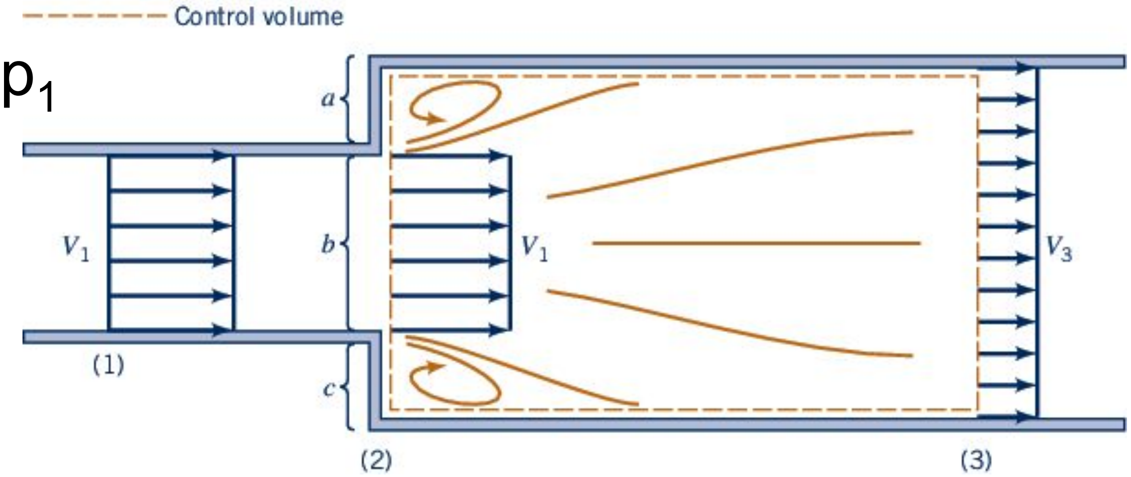
sharp-edged
K_L = 1.0

slightly rounded
K_L = 1.0

well-rounded
K_L = 1.0

Loss coefficient for a sudden expansion

$p_a=p_b=p_c=p_1$



Continuity (1): $A_1V_1=A_3V_3$

Momentum (2): $p_1A_3 - p_3A_3 = \rho A_3V_3\ (V_3 - V_1)$

Energy (3): $\frac{p_1}{\gamma}+\frac{v_1^2}{2g}=\frac{p_3}{\gamma}+\frac{v_3^2}{2g}+h_l$

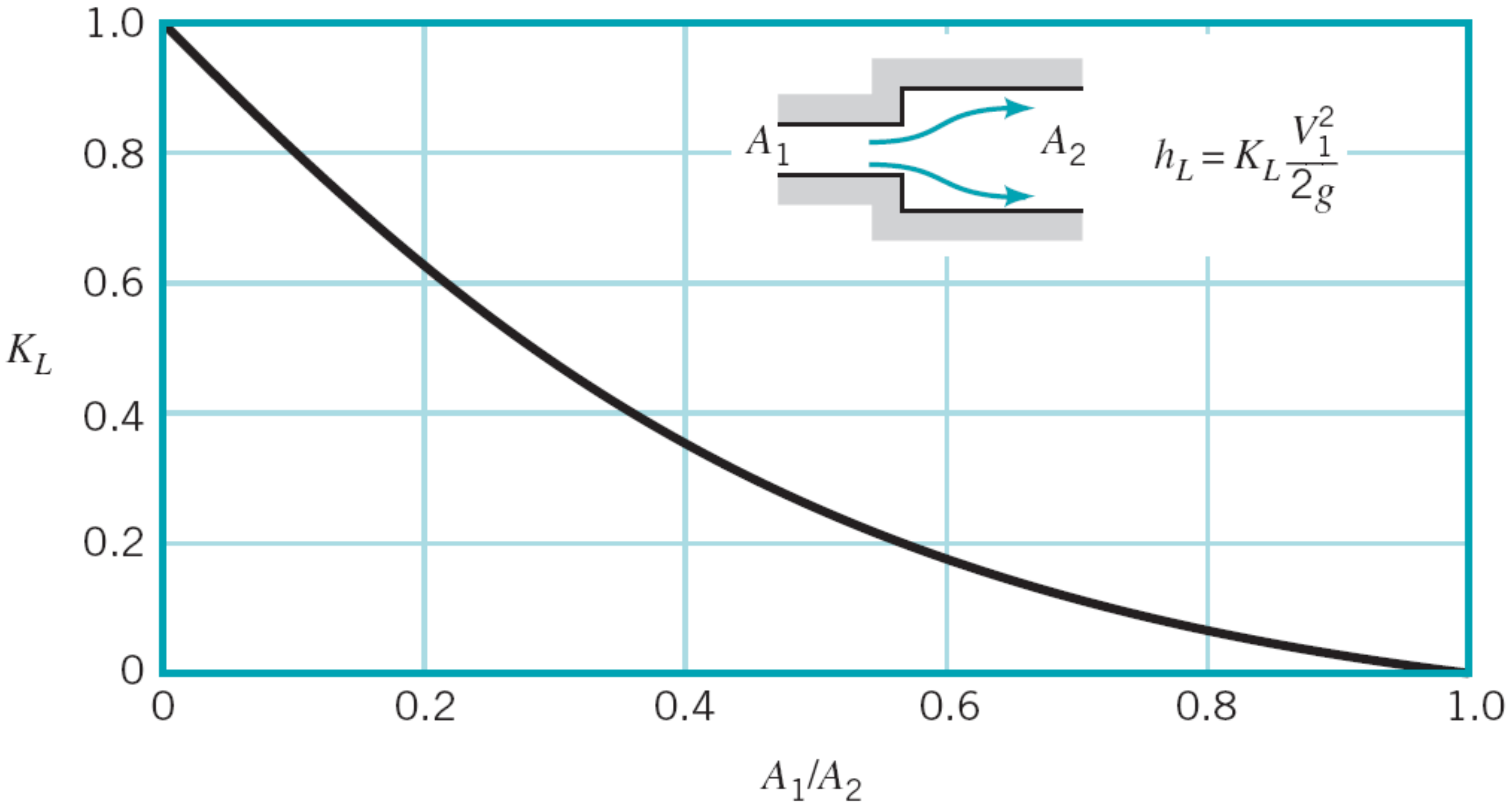
Head loss (4): $h_l=K_l\frac{v_1^2}{2g}$

$$h_l=\frac{p_1-p_3}{\gamma}+\frac{v_1^2-v_3^2}{2g}=\frac{\rho v_3(v_3-v_1)}{\gamma}+\frac{v_1^2-v_3^2}{2g} \qquad (2+3)$$

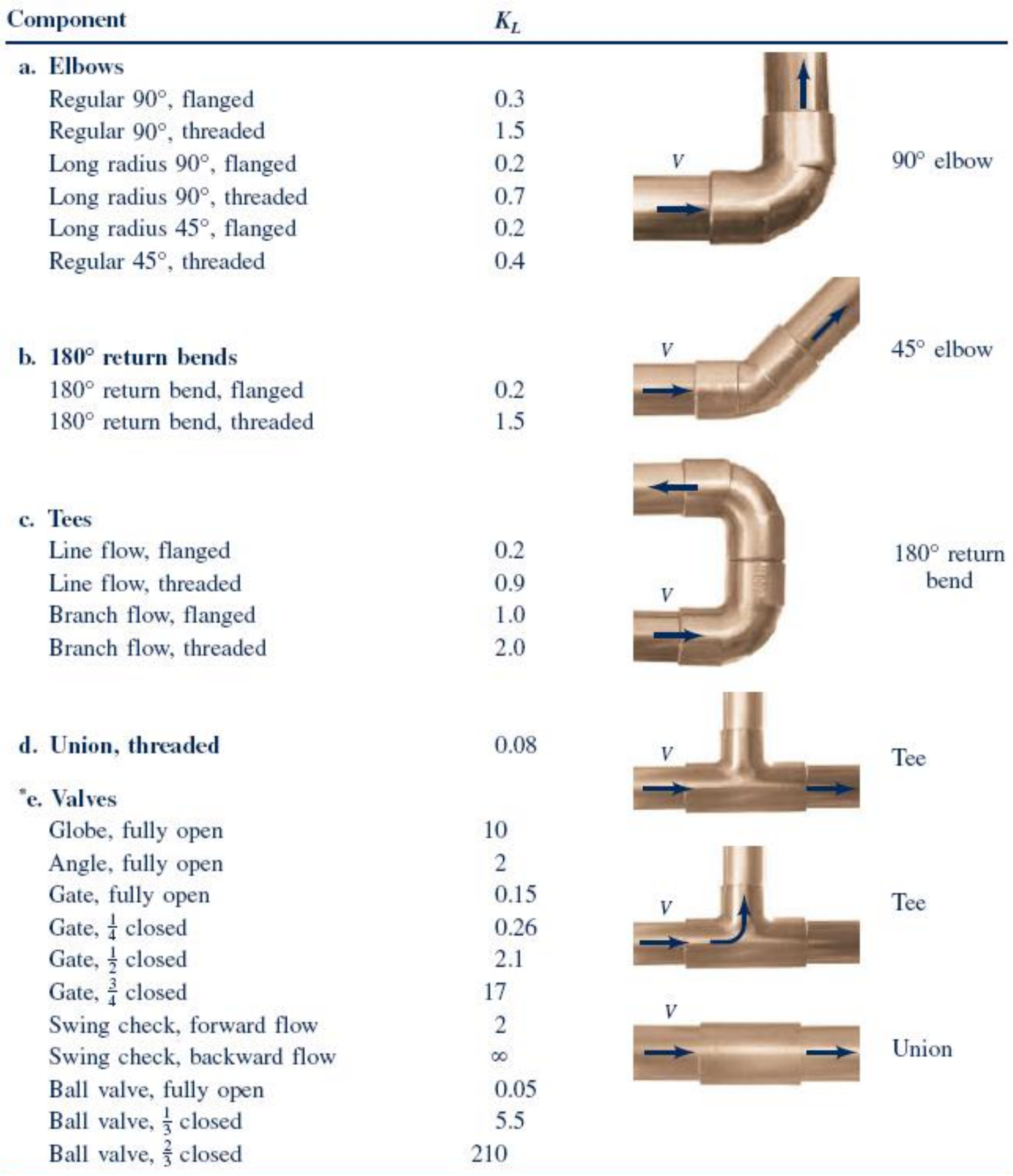
→ $$K_l=\frac{h_l*2g}{v_1^2}=\frac{2v_3^2-2v_3v_1}{v_1^2}+\frac{v_1^2-v_3^2}{v_1^2}=1+\frac{v_3^2}{v_1^2}-\frac{2v_3}{v_1} \qquad (4)$$

→ $$K_l=(1-\frac{v_3}{v_1})^2=(1-\frac{A_1}{A_3})^2 \qquad (5)$$

Loss coefficient for a sudden expansion

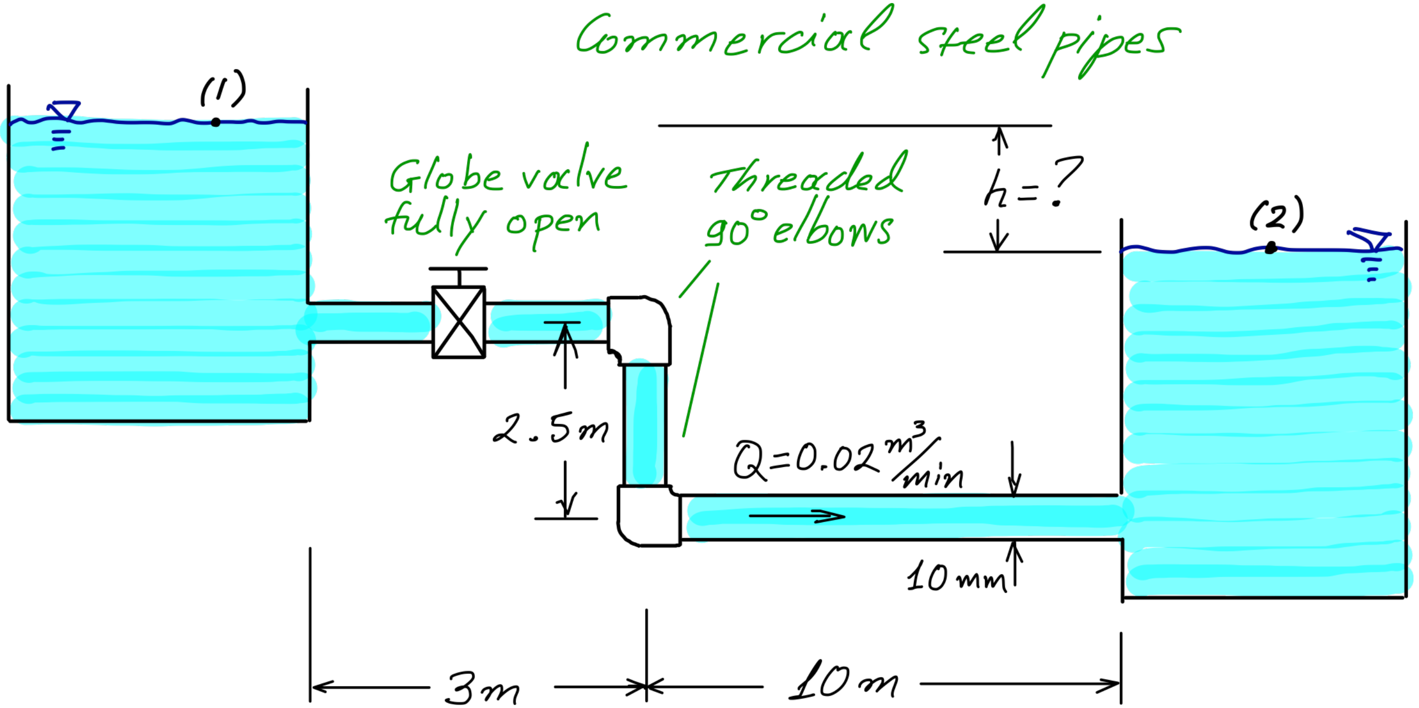


Loss coefficients for standard pipe components



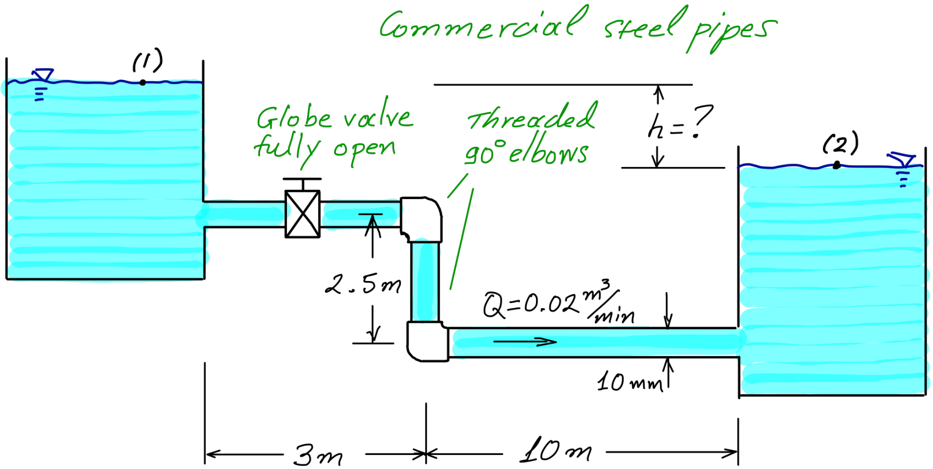
Component	K_L
a. Elbows	
Regular 90°, flanged	0.3
Regular 90°, threaded	1.5
Long radius 90°, flanged	0.2
Long radius 90°, threaded	0.7
Long radius 45°, flanged	0.2
Regular 45°, threaded	0.4
b. 180° return bends	
180° return bend, flanged	0.2
180° return bend, threaded	1.5
c. Tees	
Line flow, flanged	0.2
Line flow, threaded	0.9
Branch flow, flanged	1.0
Branch flow, threaded	2.0
d. Union, threaded	0.08
***e. Valves**	
Globe, fully open	10
Angle, fully open	2
Gate, fully open	0.15
Gate, $\frac{1}{4}$ closed	0.26
Gate, $\frac{1}{2}$ closed	2.1
Gate, $\frac{3}{4}$ closed	17
Swing check, forward flow	2
Swing check, backward flow	∞
Ball valve, fully open	0.05
Ball valve, $\frac{1}{3}$ closed	5.5
Ball valve, $\frac{2}{3}$ closed	210

Example



Energy eq: $$\cancelto{0}{\frac{P_1}{\gamma}} + \cancelto{0}{\frac{V_1^2}{2g}} + z_1 = \cancelto{0}{\frac{P_2}{\gamma}} + \cancelto{0}{\frac{V_2^2}{2g}} + z_2 + h_L$$

$$\Rightarrow \quad h = z_1 - z_2 = h_L$$

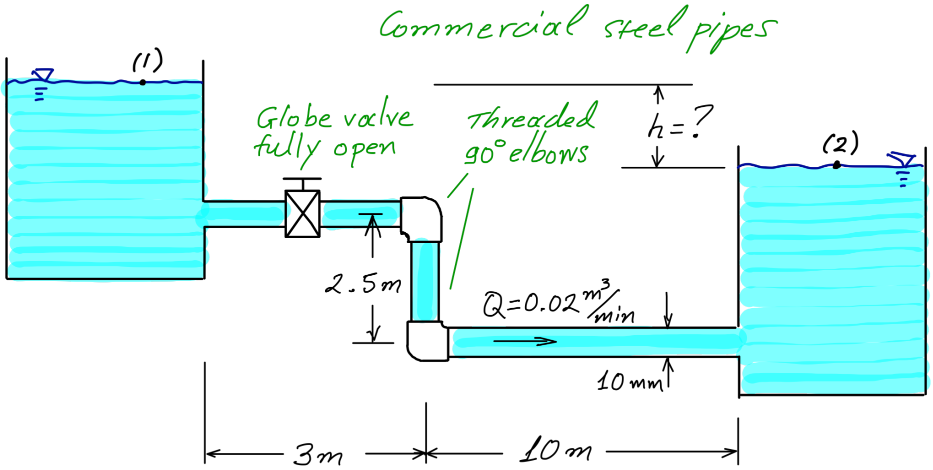


$$h_L = \text{major head losses} + \text{minor head losses}$$

$$= f\cdot\frac{l_1}{D}\frac{V^2}{2g} + f\frac{l_2}{D}\frac{V^2}{2g} + f\frac{l_3}{D}\frac{V^2}{2g}$$

$$+ K_{ent}\frac{V^2}{2g} + K_{valve}\frac{V^2}{2g} + 2\cdot K_{elbow}\frac{V^2}{2g} + K_{exit}\frac{V^2}{2g}$$

$$\Rightarrow h_L = \underbrace{f\frac{l_{total}}{D}\cdot\frac{V^2}{2g}}_{\text{major head losses}} + \underbrace{\left(K_{ent} + K_{valve} + 2K_{elbow} + K_{exit}\right)\frac{V^2}{2g}}_{\text{minor head losses}}$$



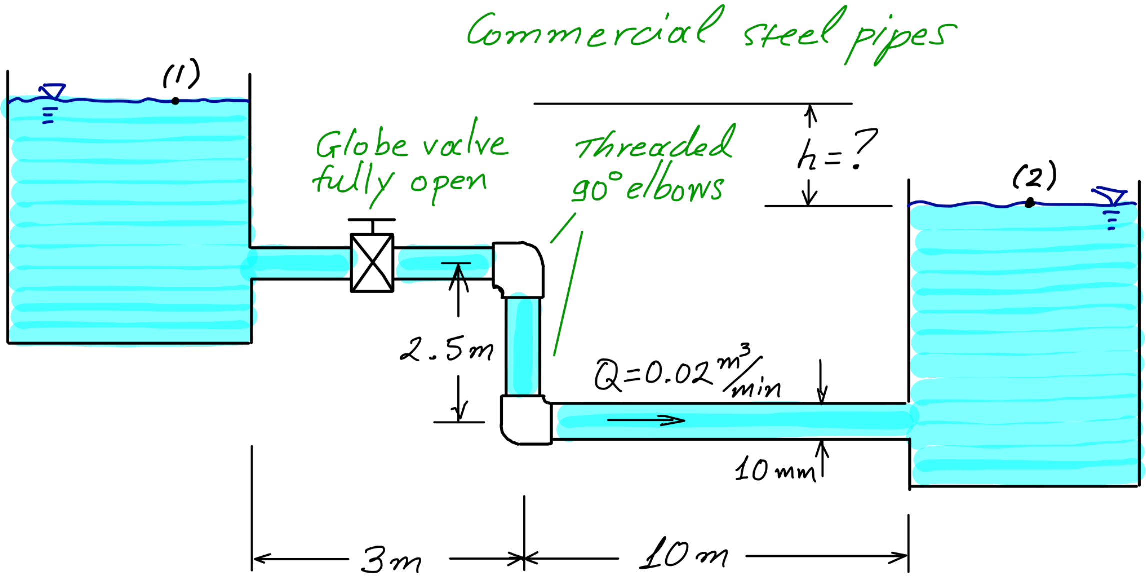
$$V = \frac{Q}{A} = \frac{0.02\ m^3/60\ s}{\pi \times 0.005^2\ m^2} = 4.24\ m/s$$

$$Re = \frac{\rho V D}{\mu} = \frac{999 \times 4.24 \times 0.01}{1.12 \times 10^{-3}} = \underline{3.7 \times 10^4}$$

From Tables: commercial steel $\varepsilon = 0.045\ mm$

$$\therefore \quad \frac{\varepsilon}{D} = \frac{0.045\ mm}{10\ mm} = \underline{0.0045}$$

From Colebrook's formula: $\underline{f = 0.032}$



From tables:

$K_{ent} = 0.5$

$K_{valve} = 10$

$K_{elbow} = 1.5$

$K_{exit} = 1.0$

$$h_L = \left[f \cdot \frac{l_{total}}{D} + K_{ent} + K_{valve} + 2\,K_{elb} + K_{exit} \right] \frac{V^2}{2g}$$

$$= \left[0.032 \times \frac{15.5}{0.1} + 0.5 + 10 + 2 \times 1.5 + 1.0 \right] \frac{4.24^2}{2 \times 9.81}\ m$$

$$\Rightarrow \quad \underline{\underline{h_L = 58.7\ m}}$$

Non-circular pipes

Define hydraulic diameter, D_h

$$D_h = \frac{4A}{P}$$ P: wetted perimeter

Check for circular pipes: $D_h = \frac{4 \cdot A}{P} = \frac{4 \cdot \frac{\pi D^2}{4}}{\pi D} = D$ ✓

Method: Work as for circular pipes using D_h as the equivalent diameter

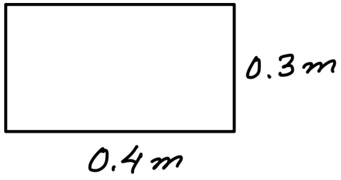
i.e., $$h_L = f \frac{\ell}{D_h} \frac{V^2}{2g}$$

$$Re_h = \frac{\rho V D_h}{\mu}$$

For laminar flow: $f = \frac{C}{Re_h}$ (C from tables)

For turbulent flow: Use Moody's chart

Example



Commercial steel

Air-conditioning duct, delivers $Q = 1\ m^3/s$ of air.

- What is the pressure drop per 10 m?
- How much more efficient is a circular duct?

$$D_h = \frac{4 \cdot A}{P} = \frac{4 \times (0.3m \times 0.4m)}{2 \times (0.3m + 0.4m)} = \underline{0.343\ m}$$

$$V = \frac{Q}{A} = \frac{1\ m^3/s}{0.3m \times 0.4m} = 8.33\ m/s$$

$$Re_h = \frac{\rho V D_h}{\mu} = \frac{1.23 \cdot 8.33 \cdot 0.343}{1.8 \times 10^{-5}} = 1.95 \times 10^5$$

$$\frac{\varepsilon}{D_h} = \frac{0.000045}{0.343} = 0.00013$$

From Moody's chart: $f = 0.0168$

$$\therefore\ \Delta p = f \cdot \frac{l}{D_h} \cdot \frac{\rho V^2}{2} = 0.0168 \times \frac{10}{0.343} \times \frac{1.23 \times 8.33^2}{2}$$

$$\Longrightarrow\ \underline{\Delta p = 20.9\ N/m^2}$$

For a circular pipe with the same area:

$$\frac{\pi D^2}{4} = 0.3\,m \times 0.4\,m = 0.12\,m^2 \quad \Rightarrow D = \sqrt{\frac{4\times 0.12}{\pi}} = 0.391\,m$$

$$Re = \frac{\rho V D}{\mu} = \frac{1.23 \times 8.33 \times 0.391}{1.8 \times 10^{-5}} = 2.23 \times 10^{5}$$

$$\frac{\varepsilon}{D} = \frac{0.000045}{0.391} = 0.00012$$

From Moody's chart: $f = 0.0165$

$$\Delta p = f\,\frac{l}{D}\,\frac{V^2}{2g} = 0.0165\,\frac{10}{0.391}\cdot\frac{8.33^2}{2\times 9.81}\ N/m^2$$

$$\Rightarrow \quad \Delta p = 18.0\ N/m^2$$

The circular pipe is $\frac{20.9 - 18.0}{20.9} = 13.8\%$ more efficient

Single pipe problems

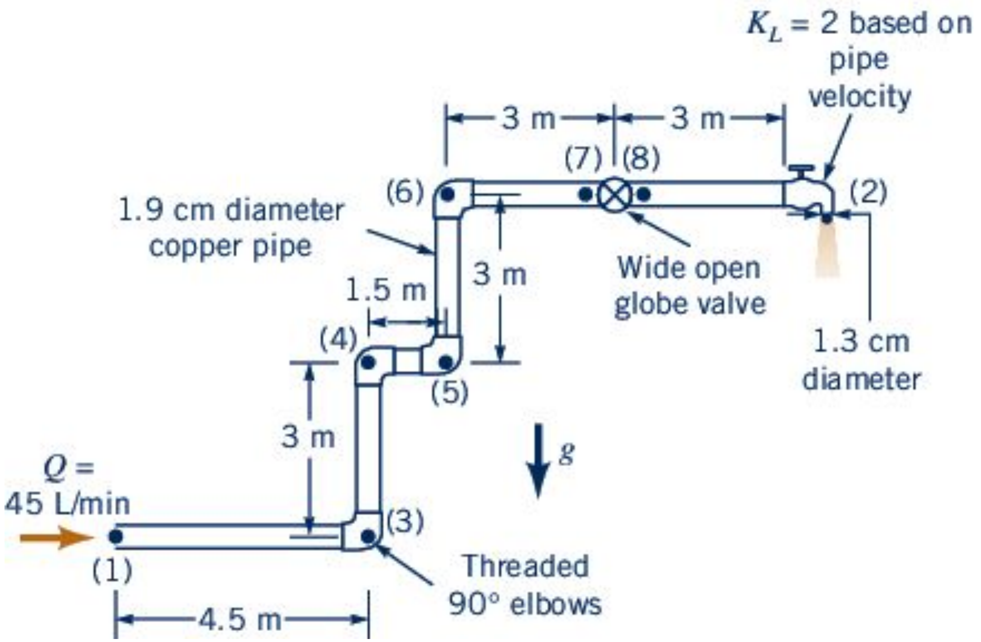
Method: Apply the **energy equation** between two convenient points in the flow system

$$h_L = \underbrace{\sum_i f_i \frac{l_i}{D_i}\frac{V_i^2}{2g}}_{\text{Major head losses}} + \underbrace{\sum_j k_{L_j}\frac{V_j^2}{2g}}_{\text{Minor head losses}}$$

Pipe Flow Type problems

Variable	Type I	Type II	Type III
a. Fluid			
Density	Given	Given	Given
Viscosity	Given	Given	Given
b. Pipe			
Diameter	Given	Given	Determine
Length	Given	Given	Given
Roughness	Given	Given	Given
c. Flow			
Flowrate or Average Velocity	Given	Determine	Given
d. Pressure			
Pressure Drop or Head Loss	Determine	Given	Given

Example - Type I problem (pressure drop unknown)



Water at 15 °C flows from the basement to the second floor and exits from a faucet. What is the pressure at point (1) if one takes into account

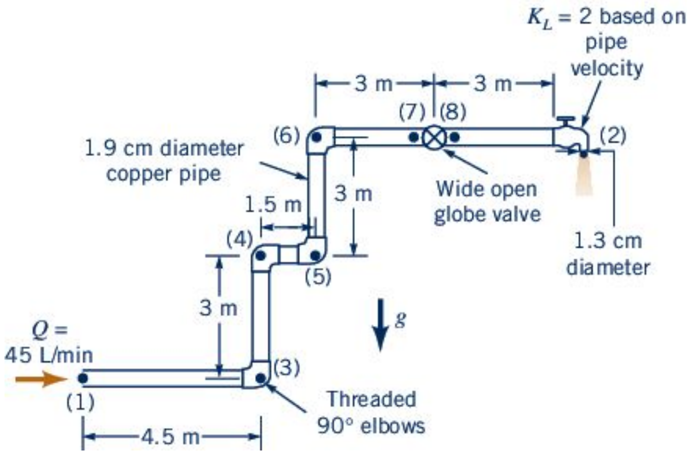
(a) no losses
(b) only major losses
(c) both major and minor losses

$$v_1 = \frac{Q}{A_1} = \frac{7.5 * 10^{-4}\, m^3/_s}{\pi(\frac{0.019)^2}{4}\, m^2} = 2.65 \text{ m/s}$$

$$v_2 = \frac{Q}{A_2} = \frac{7.5 * 10^{-4}\, m^3/_s}{\pi(\frac{0.013)^2}{4}\, m^2} = 5.65 \text{ m/s}$$

$$Re = \frac{\rho v D}{\mu} = \frac{998.2^{kg}/_{m^3} * 2.65^{m}/_s * 0.019\, m}{0.001002\, ^{Ns}/_{m^2}} = 50100$$

Example - Type I problem (pressure drop unknown)



$$\frac{p_1}{\gamma} + \alpha_1 \frac{v_1^2}{2g} + z_1 = \frac{p_2}{\gamma} + \alpha_2 \frac{v_2^2}{2g} + z_2 + h_l$$

Assume uniform velocity profile across pipe

→ $\alpha_1 = \alpha_2 = 1$

(a) No losses → $h_l = 0$

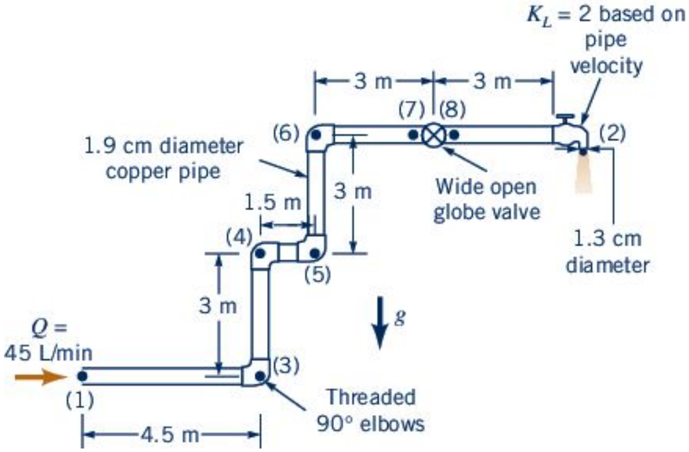
$$p_1 = \gamma(z_2 - z_1) + \frac{\rho}{2}(v_2^2 - v_1^2)$$

$$= 9.79\ {}^{kN}/_{m^3}\ 6\ m + \frac{989.2}{2}\ {}^{kg}/_{m^3} * (5.65^2\ - 2.65^2)\ {}^{m^2}/_{s^2}$$

$$= (58.74 + 12.4)\ {}^{kN}/_{m^2}$$

= 71 kPa

Example - Type I problem (pressure drop unknown)



$$\frac{p_1}{\gamma} + \alpha_1 \frac{v_1^2}{2g} + z_1 = \frac{p_2}{\gamma} + \alpha_2 \frac{v_2^2}{2g} + z_2 + h_l$$

(b) Major losses→ $h_l = f \frac{l}{D} \frac{v_1^2}{2g}$

From table: copper (drawn tubing): ε = 1.5 * 10^{-6} m → ε/D = 8 * 10^{-5}

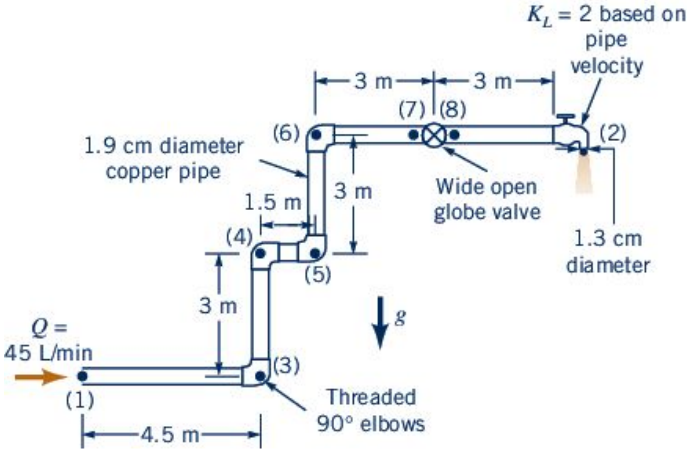
Moody`s chart with ε/D = 8 * 10^{-5} and Re=50100 → f=0.0215

$$p_1 = \gamma(z_2 - z_1) + \frac{\rho}{2}(v_2^2 - v_1^2) + \rho f \frac{l}{D} \frac{v_1^2}{2}$$

$$= 71 \text{ kPa} + 998.2\ {}^{kg}/_{m^3} * 0.0215 * (\frac{18}{0.019} m) * \frac{2.65^2}{2}\ {}^{m^2}/_{s^2}$$

= 143 kPa

Example - Type I problem (pressure drop unknown)



$$\frac{p_1}{\gamma} + \alpha_1\frac{v_1^2}{2g} + z_1 = \frac{p_2}{\gamma} + \alpha_2\frac{v_2^2}{2g} + z_2 + h_l$$

(c) Major + minor losses → $h_l = f\,\frac{l}{D}\frac{v_1^2}{2g} + \Sigma K_l\,\frac{v^2}{2g}$

From table: Loss coefficient for wide open global valve = 10

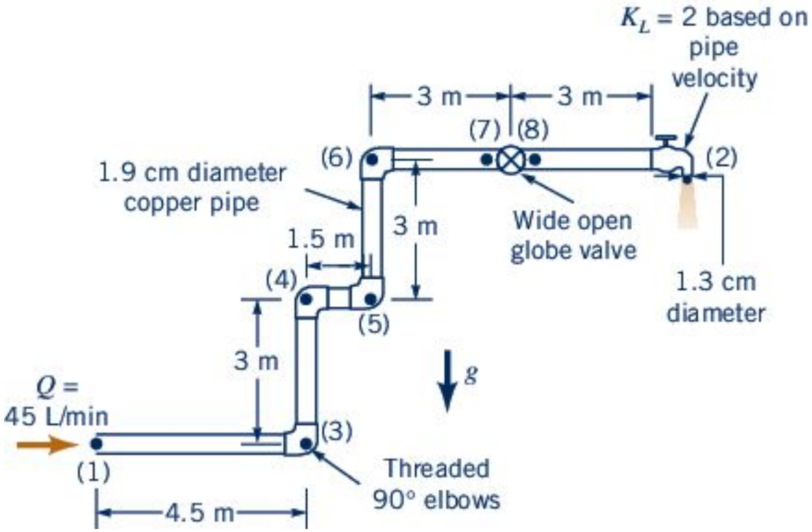
Loss coefficient for elbow = 1.5

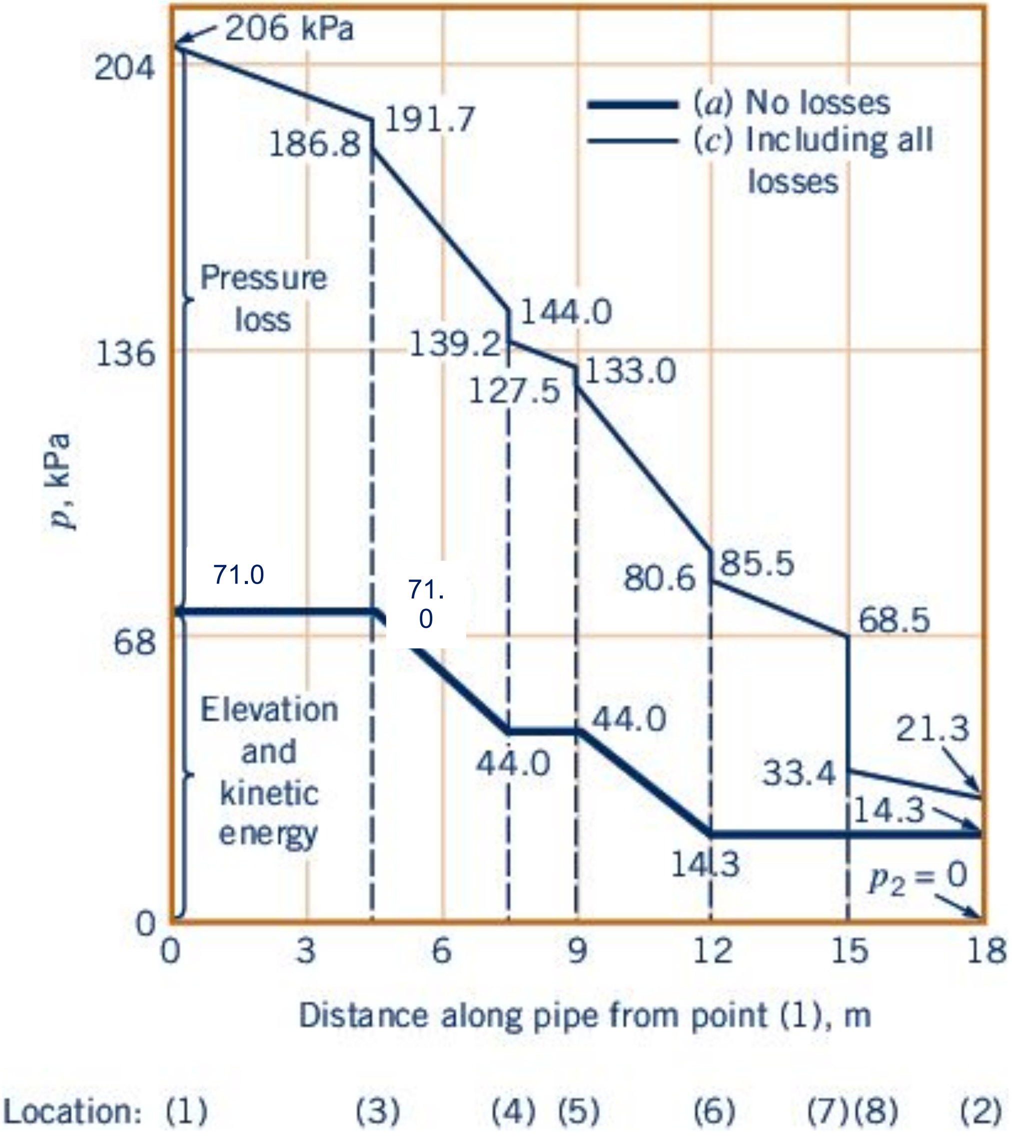
$$p_1 = \gamma(z_2 - z_1) + \frac{\rho}{2}\left(v_2^2 - v_1^2\right) + \rho f\,\frac{l}{D}\frac{v_1^2}{2} + \Sigma\rho K_l\,\frac{v^2}{2}$$

$$= 143 \text{ kPa} + 998.2\ {}^{kg}/_{m^3} * \frac{2.65^2}{2}\ {}^{m^2}/_{s^2} * (10 + 4*1.5 + 2)$$

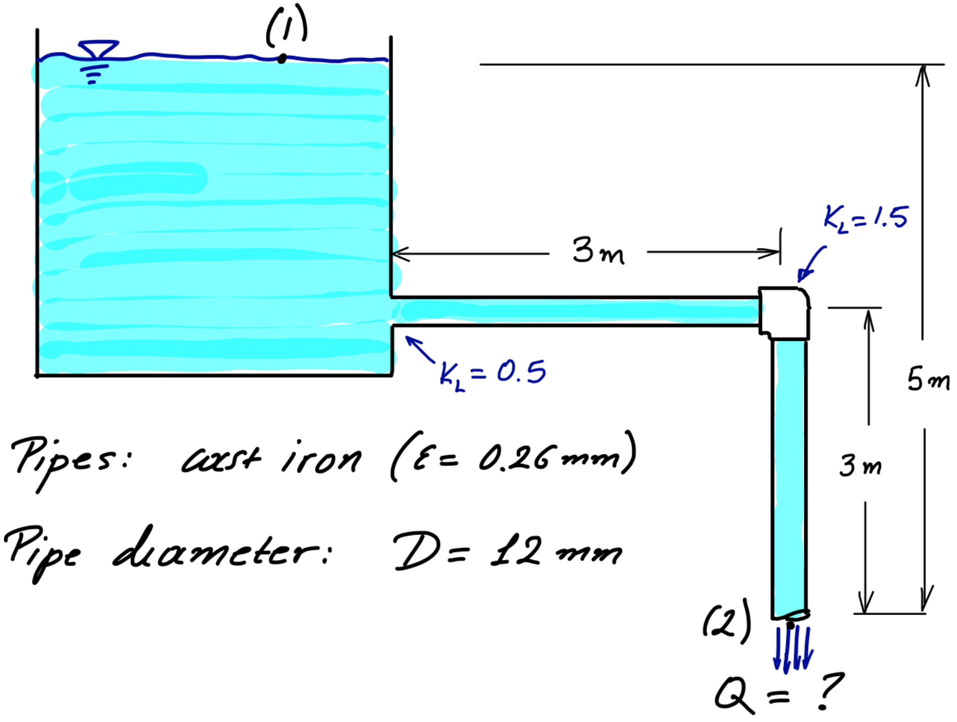
= 206 kPa

Example - Type I problem (pressure drop unknown)





Example - Type II problem (flow unknown)



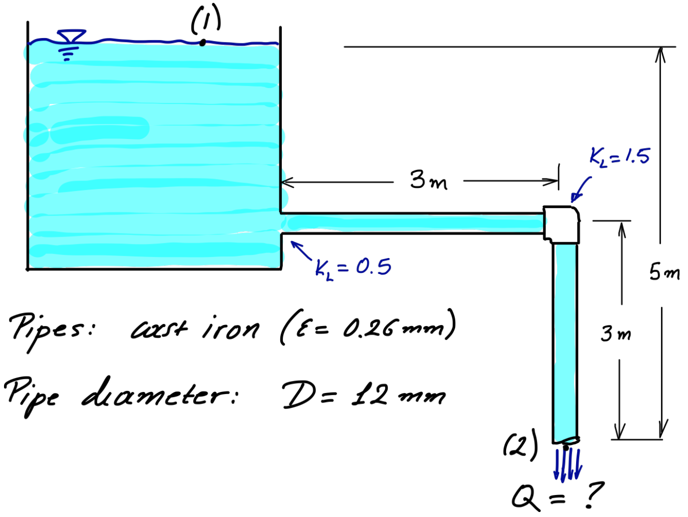
Pipes: cast iron ($\varepsilon = 0.26\,mm$)

Pipe diameter: $D = 12\ mm$

$$\cancelto{0}{\frac{p_1}{\gamma}} + \cancelto{0}{\frac{V_1^2}{2g}} + z_1 = \cancelto{0}{\frac{p_2}{\gamma}} + \frac{V_2^2}{2g} + z_2 + h_L$$

$$\Rightarrow \quad z_1 - z_2 = \frac{V_2^2}{2g} + h_L \qquad V_2 = V$$

$$h_L = f\frac{\ell}{D}\frac{V^2}{2g} + \Sigma K_L \frac{V^2}{2g} = \left(f\frac{\ell}{D} + \Sigma K_L\right)\frac{V^2}{2g}$$



Pipes: cast iron ($\varepsilon = 0.26\,mm$)

Pipe diameter: $D = 12\,mm$

$$\therefore \quad z_1 - z_2 = \left(1 + f\frac{\ell}{D} + \Sigma K_L\right)\frac{V^2}{2g}$$

$$\Rightarrow \quad 5 = \left(1 + f\frac{6}{0.012} + (0.5 + 1.5)\right)\frac{V^2}{2\times 9.81}$$

$$\Rightarrow \quad V = \sqrt{\frac{98.1}{3 + 500f}} \qquad (1)$$

$$Re = \frac{\rho V D}{\mu} = \frac{999\times V\cdot 0.012}{1.12\times 10^{-3}} = 10.7\times 10^{3}\cdot V \quad \Rightarrow \quad Re = 10'700\ V \qquad (2)$$

Procedure:
1) Assume an f value (usually $f=0.02$)
2) Calculate V (eq. 1)
3) Calculate Re (eq. 2)
4) Calculate f (Moody chart)
5) Repeat steps 2 to 4 until convergence (no change in f)

Apply procedure: 1) $f = 0.02$

2) $V = \sqrt{\dfrac{98.1}{3+500\times 0.02}} = 2.75 \text{ m/s}$

3) $Re = 10700 \times 2.75 = 2.94 \times 10^4$

4) $\dfrac{\varepsilon}{D} = \dfrac{0.26 \text{ mm}}{12 \text{ mm}} = 0.0217$

From Moody's chart: $f = 0.050$ $(\neq 0.02)$

Repeat procedure: 1) $f = 0.050$

2) $V = \sqrt{\dfrac{98.1}{3+500\times 0.05}} = 1.87 \text{ m/s}$

3) $Re = 10700 \times 1.87 = 2.00 \times 10^4$

4) From Moody's chart: $f = 0.051$ $(\neq 0.050)$

Repeat: 1) $f = 0.051$

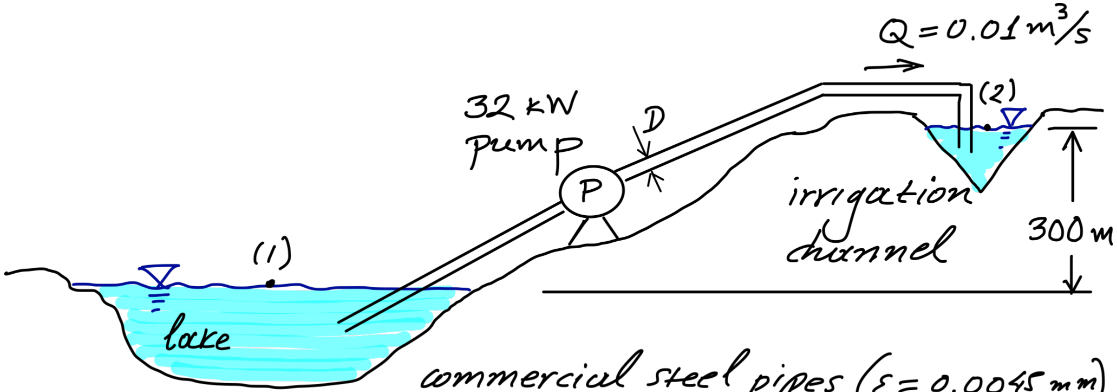
2) $V = \sqrt{\dfrac{98.1}{3+500\times 0.051}} = 1.86 \text{ m/s}$

3) $Re = 10700 \times 1.86 = 1.99 \times 10^4$

4) From Moody's chart: $\underline{f = 0.051}$ Converged!

$$Q = A \cdot V = \pi \times 0.006^2 \text{ m}^2 \times 1.86 \text{ m/s} = \underline{\underline{2.10 \times 10^{-4} \text{ m/s}}}$$

Example - Type III problem (diameter unknown)



commercial steel pipes ($\varepsilon = 0.0045\ mm$)

Pipe length: 2 km

Neglect minor losses

$$\text{Energy: } \cancelto{0}{\frac{p_1}{\gamma}} + \cancelto{0}{\frac{V_1^2}{2g}} + z_1 + h_p = \cancelto{0}{\frac{p_2}{\gamma}} + \cancelto{0}{\frac{V_2^2}{2g}} + z_2 + h_L$$

$$\Rightarrow \quad h_p - (z_2 - z_1) = f\,\frac{\ell}{D}\,\frac{V^2}{2g}$$

$$h_p = \frac{\dot{W}}{\gamma Q} = \frac{32'000\ N\cdot m/s}{9800\ N/m^3 \cdot 0.01\ m^3/s} = 326\ m$$

$$\Rightarrow \quad 326\ m - 300\ m = f \cdot \frac{2000}{D} \cdot \frac{\left(\frac{Q}{\pi D^2/4}\right)^2}{2 \times 9.81}$$

$$\Rightarrow \quad 0.2044 = \frac{f \cdot Q^2}{D^5} = \frac{f \cdot (0.01)^2}{D^5}$$

$$\Rightarrow \quad \underline{D = \left(4.89 \times 10^{-4} \times f\right)^{1/5}} \qquad (1)$$

$$Re = \frac{\rho V D}{\mu} = \frac{\rho \frac{Q}{\pi D^2/4} D}{\mu} = \frac{4 \rho Q}{\pi \mu D} = \frac{4 \times 999 \times 0.01}{\pi \times 1.12 \times 10^{-3} \cdot D}$$

$$\Rightarrow \quad \underline{\underline{Re = \frac{1.14 \times 10^{4}}{D}}} \quad (2) \qquad\qquad \underline{\underline{\frac{\varepsilon}{D} = \frac{0.045 \times 10^{-3}}{D}}} \quad (3)$$

Iterative procedure:

Assume $f = 0.02$

$(1) \Rightarrow D = 0.0996 \text{ m}$

$(2) \Rightarrow Re = 1.145 \times 10^{5}$

$(3) \Rightarrow \varepsilon/D = 0.000452$

Moody's chart $\Rightarrow f = 0.0198$ (close to 0.02)

Repeat: $(1) \Rightarrow D = 0.0994 \text{ m}$

$(2) \Rightarrow Re = 1.15 \times 10^{5}$

$(3) \Rightarrow \varepsilon/D = 0.000453$

Moody's chart $\Rightarrow \underline{\underline{f = 0.0198}}$ Converged

$$\therefore \quad D = 0.0994 \text{ m} \quad \text{or} \quad \underline{\underline{D = 9.94 \text{ cm}}}$$